Q1

$$g(x,y) = \begin{cases} y e^{-1/x^2}, & \text{for } x \neq 0, \\ y, & \text{for } x = 0 \end{cases}$$

(a) Calculate $\lim\limits_{(x,y)\to(0,0)} g(x,y)$

Since $\frac{-1}{x^2} \leq 0$ then $e^{\frac{-1}{x^2}} \leq 1$.

So $|g(x,y)| \leq |y \cdot 1| = |y|$.

So $\lim\limits_{(x,y)\to(0,0)} |g(x,y)| \leq \lim\limits_{(x,y)\to(0,0)} |y| = 0$.

So $\lim\limits_{(x,y)\to(0,0)} |g(x,y)| = 0$.

So $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0$.

(b) The function $\frac{-1}{x^2}$ is continuous for $x \neq 0$

since x^2 is a polynomial.

The function y is continuous for $y \in \mathbb{R}$.

Since e^z is continuous for $z \in \mathbb{R}$ then

$y e^{\frac{-1}{x^2}}$ is continuous for $(x,y) \in \mathbb{R}^2$ with $x \neq 0$.

Since $\lim\limits_{x\to 0} e^{\frac{-1}{x^2}} = \lim\limits_{z\to -\infty} e^z = 0$ then

$$\lim_{\substack{(x,y)\to(0,a)\\ y=a}} g(x,y) = \lim_{x\to 0} a e^{-\frac{1}{x^2}} = a\cdot 0 = 0$$

and

$$\lim_{\substack{(x,y)\to(0,a)\\ x=0}} g(x,y) = \lim_{y\to a} y = a.$$

So $g(x,y)$ is <u>not</u> continuous at $(0,a)$ for $a\neq 0$.

By part (a) $\lim_{(x,y)\to(0,0)} g(x,y) = 0$,

and $g(0,0) = 0$, so $g(x,y)$ is continuous at $(0,0)$.

(c) $$\left.\frac{\partial g}{\partial x}\right|_{(x,y)=(0,0)} = \lim_{h\to 0} \frac{g(0+h,0)-g(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0.$$

$$\left.\frac{\partial g}{\partial y}\right|_{(x,y)=(0,0)} = \lim_{h\to 0} \frac{g(0,0+h)-g(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h} = \lim_{h\to 0} 1 = 1.$$

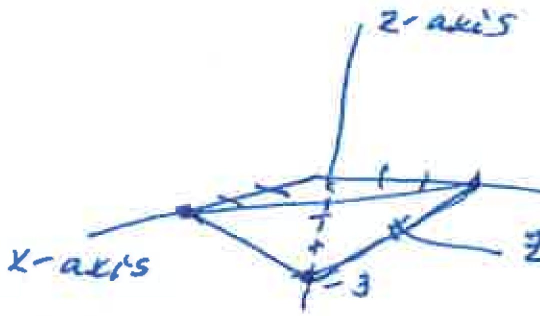
Q1 (a) Minimise $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$
with constraint $g=0$ where
$$(g(x,y)-z) = x+y-z-3.$$

Solution Using Lagrange multipliers $\vec{\nabla} f = \lambda \vec{\nabla} g$
gives $(x, y, z) = \lambda(1,1,-1)$ so that

$x = \lambda$
$y = \lambda$
$z = -\lambda$
and $x+y-z-3=0$.

So $\lambda + \lambda - (-\lambda) - 3 = 0$
So $3\lambda = 3$
So $\lambda = 1$.

So the critical point is $(x,y,z) = (1,1,-1)$



$z = x+y-3$ is the plane containing this triangle

From the picture, there is a unique point on the plane closest to the origin

(b) For the general case,
minimise $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$ with constraint $(g(x,y)-z) = 0$ to get equations

$$x = \lambda \frac{\partial g}{\partial x}$$
$$y = \lambda \frac{\partial g}{\partial y}$$
$$z = \lambda \frac{\partial g}{\partial z}$$
$$(g(x,y)-z) = 0.$$

One could write

$$y \frac{\partial g}{\partial x} = x \frac{\partial g}{\partial y}, \quad y \frac{\partial g}{\partial z} = z \frac{\partial g}{\partial y}$$
$$z \frac{\partial g}{\partial x} = x \frac{\partial g}{\partial z}$$
$$(g(x,y)-z) = 0.$$

Q3

(a) $\vec{c}(t) = (2t, t^2, \log t)$.

$$\vec{v} = \frac{d\vec{c}}{dt} = \left(2, 2t, \frac{1}{t}\right) \text{ and } \vec{a} = \frac{d\vec{v}}{dt} = (0, 2, -t^{-2})$$

Then

$$\left|\frac{d\vec{c}}{dt}\right| = \sqrt{2^2 + (2t)^2 + \frac{1}{t^2}} = \sqrt{4 + 4t^2 + \frac{1}{t^2}}$$

$$= \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\frac{(2t^2+1)^2}{t^2}} = \frac{2t^2+1}{t}$$

So

$$\vec{T} = \frac{\frac{d\vec{c}}{dt}}{\left|\frac{d\vec{c}}{dt}\right|} = \frac{t}{2t^2+1}\left(2, 2t, \frac{1}{t}\right) = \frac{1}{2t^2+1}(2t, 2t^2, 1)$$

(b)

$$\frac{d\vec{T}}{dt} = \frac{-4t}{(2t^2+1)^2}(2t, 2t^2, 1) + \frac{1}{2t^2+1}(2, 4t, 0)$$

$$= \frac{1}{(2t^2+1)^2}\left((-8t^2, -8t^3, -4t) + (4t^2+2, 8t^3+4t, 0)\right)$$

$$= \frac{1}{(2t^2+1)^2}(2-4t^2, 4t, -4t)$$

Then

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{(2t^2+1)^2}\sqrt{(2-4t^2)^2 + (4t)^2 + (4t)^2}$$

$$= \frac{1}{(2t^2+1)^2}\sqrt{4 - 8t^2 + 16t^4 + 16t^2 + 16t^2}$$

$$= \frac{1}{(2t^2+1)^2}\sqrt{4+24t^2+16t^4}$$

$$= \frac{2}{(2t^2+1)^2}\sqrt{1+6t^2+4t^4}$$

The curvature at $t=1$ is:

$$\kappa(1) = \frac{\left|\frac{d\vec{T}}{dt}\right|_{t=1}}{\left|\frac{d\vec{c}}{dt}\right|_{t=1}} = \frac{\frac{2}{(2+1)^2}\sqrt{1+6+4}}{\frac{2+1}{1}} = \frac{2\sqrt{11}}{27}$$

(c) By definition, $\vec{N} = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$.

Since $\vec{T}$ is a unit vector $\vec{T}\cdot\vec{T} = |\vec{T}|^2 = 1^2 = 1$.

So $$\frac{d(\vec{T}\cdot\vec{T})}{dt} = \vec{T}\cdot\frac{d\vec{T}}{dt} + \frac{d\vec{T}}{dt}\cdot\vec{T} = 2\,\vec{T}\cdot\frac{d\vec{T}}{dt}$$

$$= 2(\vec{T}\cdot\vec{N})\left|\frac{d\vec{T}}{dt}\right|$$

and $$\frac{d(\vec{T}\cdot\vec{T})}{dt} = \frac{d1}{dt} = 0.$$

So $\vec{T}\cdot\vec{N} = 0$ and $\vec{T}$ and $\vec{N}$ are perpendicular.

So $|\vec{a}|^2 = \vec{a}\cdot\vec{a} = (a_T\vec{T} + a_N\vec{N})\cdot(a_T\vec{T} + a_N\vec{N})$

$$= a_T^2\, \vec{T}\cdot\vec{T} + a_T a_N \vec{T}\cdot\vec{N} + a_N a_T \vec{N}\cdot\vec{T} + a_N^2\, \vec{N}\cdot\vec{N}$$

$$= a_T^2\, |\vec{T}|^2 + a_T a_N \cdot 0 + a_N a_T \cdot 0 + a_N^2\, |\vec{N}|^2$$

$$= a_T^2 \cdot 1 + a_N^2 \cdot 1 = a_T^2 + a_N^2.$$

So $|\vec{a}| = \sqrt{a_T^2 + a_N^2}.$

(d) The component of $\vec{a}$ in the direction of $\vec{T}$

is $a_T = |\vec{a}| \cos\theta = \dfrac{\vec{a}\cdot\vec{T}}{|\vec{T}|} = (0,2,1)\cdot\frac{1}{3}(2,2,1)$

$$= \frac{1}{3}(0+4-1) = \frac{3}{3} = 1.$$

Since $|\vec{a}| = |(0,2,-1)| = \sqrt{2^2+1^2} = \sqrt{5}$ and

$5 = |\vec{a}|^2 = a_T^2 + a_N^2 = 1^2 + a_N^2$ then $a_N = 2$.

So $\vec{a} = a_T \vec{T} + a_N \vec{N}$ with $a_T = 1$ and $a_N = 2$.

Q4 (a) $\text{curl}(\text{grad} f) = \vec{\nabla} \times (\vec{\nabla} f)$

$$= \vec{\nabla} \times \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{aligned} &\hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) \\ &-\hat{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) \\ &+\hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) \end{aligned}$$

$$= \hat{i}\cdot 0 - \hat{j}\cdot 0 + \hat{k}\cdot 0 = 0$$

where, since f is C^2 then $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$ and $\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}$.

(b) $\text{div}(\text{curl}(\vec{F})) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F})$.

Letting $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then

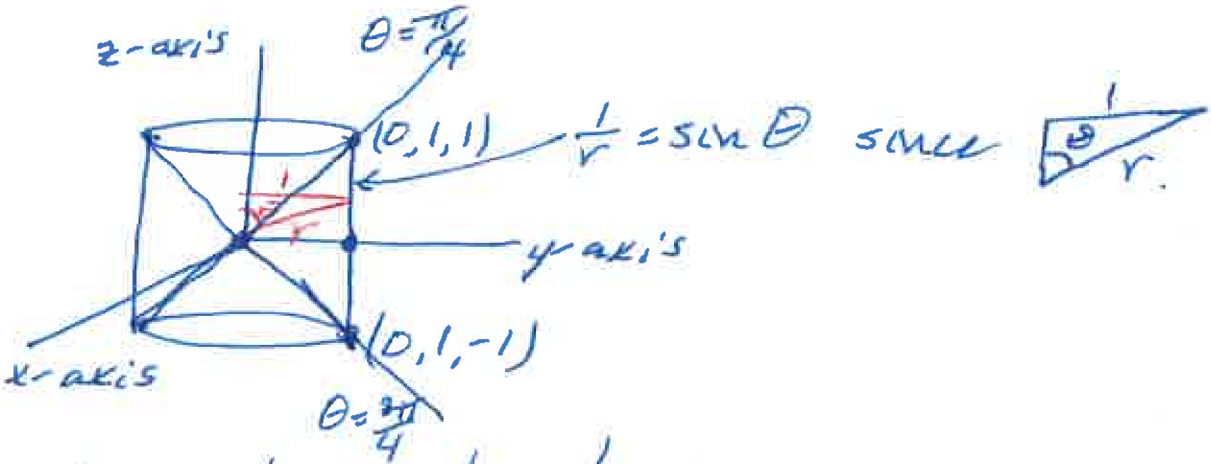
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{aligned} &\hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \\ &-\hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) \\ &+\hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right). \end{aligned}$$

So, using that $\vec{F}$ is C^2,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0.$$

Q5



In spherical coordinates:

$$\text{Volume} = \iiint_V dV = \iiint_V r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{3\pi}{4}} \int_{r=0}^{r=\frac{1}{\sin\theta}} r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{3\pi}{4}} \frac{r^3}{3}\sin\theta \Big]_{r=0}^{r=\frac{1}{\sin\theta}} d\theta\, d\varphi$$

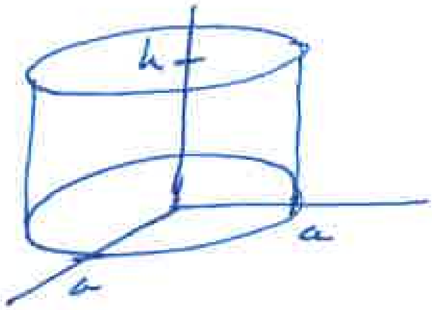
$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{3\pi}{4}} \frac{1}{3}\,\frac{1}{\sin^2\theta}\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \frac{1}{3}\cot\theta \Big]_{\theta=\frac{\pi}{4}}^{\theta=\frac{3\pi}{4}} d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \frac{1}{3}(1-(-1))\, d\varphi = \frac{2}{3}\varphi\Big]_{\varphi=0}^{\varphi=2\pi}$$

$$= \frac{4\pi}{3}.$$

Q6



height h
constant density μ
total mass M.

$$M = \mu(\text{volume}) = \mu\pi a^2 h.$$

Moment of inertia about z-axis $= \iiint_V \mu(x^2+y^2)\,dV$

$$= \int_{z=0}^{h} \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=a} \mu\rho^2\,\rho\,d\rho\,d\varphi\,dz$$

$$= \int_{z=0}^{z=h} \int_{\varphi=0}^{\varphi=2\pi} \mu\frac{\rho^4}{4}\Big]_{\rho=0}^{\rho=a} d\varphi\,dz$$

$$= \int_{z=0}^{z=h} \mu\frac{a^4}{4}\,\varphi\Big]_{\varphi=0}^{\varphi=2\pi} dz$$

$$= \int_{z=0}^{z=h} \frac{\mu a^4\pi}{2}\,dz = \frac{\mu a^4\pi}{2}\cdot h$$

$$= \frac{\mu a^4\pi}{2}\;\frac{M}{\mu\pi a^2} = \frac{Ma^2}{2}.$$

$\underline{Q7}$ $\vec{F} = 2x\hat{i} - 4yz\hat{j} - (4y^2z - 1)\hat{k}$.

(a) Find f such that $\vec{F} = \vec{\nabla} f$.

$\underline{\text{Guess:}}$ $f = x^2 - 2y^2z + z$

$\underline{\text{Check:}}$ $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= (2x - 0 + 0)\hat{i} + (0 - 4yz^2 + 0)\hat{j} + (0 - 4y^2z + 1)\hat{k}$$

$$= 2x\hat{i} - 4yz^2\hat{j} - (4y^2z - 1)\hat{k} = \vec{F}.$$

So $\vec{F}$ is conservative.

(b) Since $\vec{F}$ is conservative

$$\text{Work} = \int_C \vec{F} \cdot d\vec{s} = f\left(\begin{smallmatrix}\text{end}\\ \text{point}\end{smallmatrix}\right) - f\left(\begin{smallmatrix}\text{initial}\\ \text{point}\end{smallmatrix}\right).$$

The endpoint of C is

$$(2\cos 2\pi, 2\sin 2\pi, 2\pi) = (2, 0, 2\pi)$$

The initial point of C is

$$(2\cos 0, 2\sin 0, 0) = (2, 0, 0).$$

So

$$\text{Work} = \int_C \vec{F} \cdot d\vec{s} = f(2, 0, 2\pi) - f(2, 0, 0)$$

$$= (4 - 0 + 2\pi) - (4 - 0 + 0) = 2\pi.$$

Q8 Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where S is the surface of the ball

$$z = (1-x^2-y^2)e^{1-x^2-3y^2} \quad \text{for} \quad z \geq 0$$

and

$$\vec{F} = (e^y \cos z,\ (x^3+1)^{\frac{1}{2}} \sin z,\ x^2+y^2+3)$$

(Hint: Use the divergence theorem).

Make a solid region V bounded by S and the plane $z=0$.

The base has normal vector $-\hat{k}$ (oriented outwards).

So

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial V} \vec{F}\cdot d\vec{S} - \iint_{base} \vec{F}\cdot d\vec{S}$$

$$= \iiint_V (\vec{\nabla}\cdot\vec{F})\,dV - \iint_{base} \vec{F}\cdot d\vec{S}, \quad \text{by the Divergence theorem}$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz - \iint_{base} \vec{F}\cdot(-\hat{k})\,dx\,dy,$$

where $F_1 = e^y \cos z$, $F_2 = (x^3+1)^{\frac{1}{2}} \sin z$, $F_3 = x^2+y^2+3$

$$= \iiint_V (0+0+0)\,dx\,dy\,dz - \iint_{base} -(x^2+y^2+3)\,dx\,dy$$

$$= 0 - \iint_{base} -(r^2+3)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (r^2+3)\, r\, dr\, d\theta,$$ since the base is a circle of radius 1,

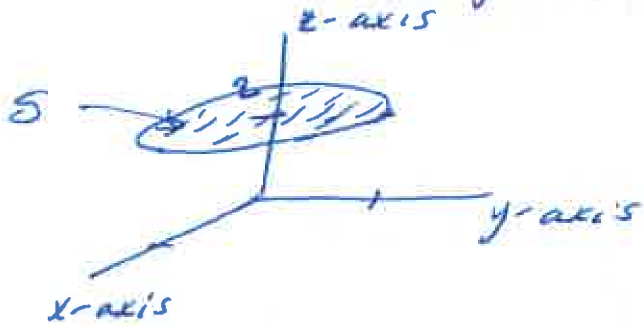
$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (r^3+3r)\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left.\frac{r^4}{4}+\frac{3r^2}{2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{4}+\frac{3}{2}-(0+0)\right) d\theta$$

$$= \left.\frac{7}{4}\theta\right]_{\theta=0}^{\theta=2\pi} = \frac{7}{4}2\pi = \frac{7}{2}\pi.$$

Q9 (a) S has $x^2+y^2 \le 9$ in the plane $z=2$



The normal vector is $\hat{k}$.

So $d\vec{S} = \hat{k}\, dx\, dy$.

$\vec{F} = z\hat{\imath} + x\hat{\jmath} + y\hat{k}$. So

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \begin{matrix} \hat{\imath}(1-0) \\ -\hat{\jmath}(0-1) \\ +\hat{k}(1-0) \end{matrix} = \hat{\imath} + \hat{\jmath} + \hat{k}.$$

So

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint_S (\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot \hat{k}\, dx\, dy$$

$$= \iint_S dx\, dy = \begin{pmatrix} \text{area of} \\ \text{circle of} \\ \text{radius } 3 \end{pmatrix} = \pi \cdot 3^2 = 9\pi.$$

(b) Using Stokes theorem,

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s} = \int_C (z\hat{\imath} + x\hat{\jmath} + y\hat{k}) \cdot \frac{d\vec{c}}{dt}\, dt$$

where $\vec{c}(t) = (3\cos t, 3\sin t, 2)$ for $0 \le t \le 2\pi$ is the curve that forms the boundary of S oriented counterclockwise.

Q9 (2)

So

$$\frac{d\vec{c}}{dt} = (-3\sin t, 3\cos t, 0) \text{ and}$$

$$\int_c (z\hat{\imath} + x\hat{\jmath} + y\hat{k}) \cdot \frac{d\vec{c}}{dt}\, dt$$

$$= \int_c (2\hat{\imath} + 3\cos t\,\hat{\jmath} + 3\sin t\,\hat{k}) \cdot (-3\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath})\, dt$$

$$= \int_{t=0}^{t=2\pi} (-6\sin t + 9\cos^2 t)\, dt$$

$$= \int_{t=0}^{t=2\pi} \left(-6\sin t + \frac{9}{2}(2\cos^2 t - 1) + \frac{9}{2}\right) dt$$

$$= \int_{t=0}^{t=2\pi} \left(-6\sin t + \frac{9}{2}\cos 2t + \frac{9}{2}\right) dt$$

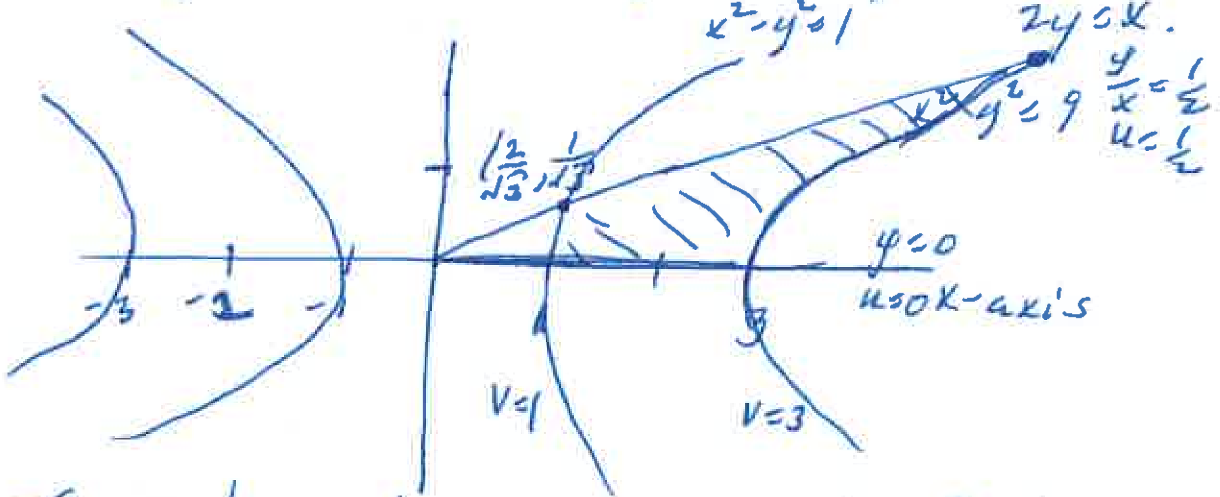
$$= 6\cos t + \frac{9}{2}\cdot\frac{1}{2}\sin 2t + \frac{9}{2}t \Big]_{t=0}^{t=2\pi}$$

$$= 6\cdot 1 + \frac{9}{4}\cdot 0 + \frac{9}{2}\cdot 2\pi - \left(6\cdot 1 + \frac{9}{4}\cdot 0 + 0\right)$$

$$= 6 + 9\pi - 6 = 9\pi.$$

Q10 (a) Graph the region bounded by

$x^2-y^2=1,\quad x^2-y^2=9,\quad y=0,\quad 2y=x.$



The intersection of $2y=x$ and $x^2-y^2=1$ is at

$$(2y)^2-y^2=1$$
$$3y^2=1$$
$$y=\frac{1}{\sqrt3}$$
$$x=\frac{2}{\sqrt3}$$

The intersection of $2y=x$ and $x^2-y^2=9$ is at

$$(2y)^2-y^2=9$$
$$3y^2=9$$
$$y^2=3$$
$$y=\sqrt3$$
$$x=2\sqrt3$$

(b) $u=\frac{y}{x}\quad v=x^2-y^2\quad \frac{v}{x^2}=1-\frac{y^2}{x^2}=1-u^2$

$$\det\frac{\partial(u,v)}{\partial(x,y)}=\det\begin{pmatrix}-\frac{y}{x^2} & \frac{1}{x}\\ 2x & -2y\end{pmatrix}=\frac{2y^2}{x^2}-2=\frac{2y^2-2x^2}{x^2}=\frac{2v}{x^2}=2(1-u^2)$$

So

$$\iint_D dy\,dx=\iint_D \det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)du\,dv=\iint\frac{x^2}{2y^2-2x^2}\,du\,dv$$
$$=\iint\frac{1}{2v}x^2\,du\,dv=\iint 2(1-u^2)\,du\,dv.$$

$$= \frac{1}{2}\int_{v=1}^{v=3}\int_{u=0}^{u=\frac{1}{2}} \frac{1}{1-u^2}\,du\,dv$$

$$= \frac{1}{2}\int_{v=1}^{v=3}\int_{u=0}^{u=\frac{1}{2}} \frac{1}{2}\left(\frac{1}{1+u}+\frac{1}{1-u}\right)du\,dv$$

$$= \frac{3}{2}\cdot\frac{1}{2}\Big(\log(1+u)+\log(1-u)\Big)\Big]_{u=0}^{u=\frac{1}{2}}$$

$$= \frac{3}{4}\log\frac{3}{2}.$$

Q11 For spherical coordinates

$$x = r\cos\varphi \sin\theta$$
$$y = r\sin\varphi \sin\theta$$
$$z = r\cos\theta$$

So

$$\hat{r} = \frac{1}{h_r}\left(\cos\varphi \sin\theta\, \hat{\imath} + \sin\varphi \sin\theta\, \hat{\jmath} + \cos\theta\, \hat{k}\right) \text{ with}$$

$$h_r = \sqrt{\cos^2\varphi \sin^2\theta + \sin^2\varphi \sin^2\theta + \cos^2\theta}$$
$$= \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1} = 1,$$

$$\hat{\theta} = \frac{1}{h_\theta}\left(r\cos\varphi \cos\theta\, \hat{\imath} + r\sin\varphi \cos\theta\, \hat{\jmath} - r\sin\theta\, \hat{k}\right),$$

$$h_\theta = \sqrt{r^2\cos^2\varphi \cos^2\theta + r^2\sin^2\varphi \cos^2\theta + r^2\sin^2\theta}$$
$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = r,$$

$$\hat{\varphi} = \frac{1}{h_\varphi}\left(-r\sin\varphi \sin\theta\, \hat{\imath} + r\cos\varphi \sin\theta\, \hat{\jmath} + 0\, \hat{k}\right),$$

$$h_\varphi = \sqrt{r^2\sin^2\varphi \sin^2\theta + r^2\cos^2\varphi \sin^2\theta}$$
$$= \sqrt{r^2\sin^2\theta} = r\sin\theta$$

So

$h_r = 1$, $h_\theta = r$ and $h_\varphi = r\sin\theta$

(b) Then

$$\hat{r} = \cos\varphi \sin\theta\, \hat{\imath} + \sin\varphi \sin\theta\, \hat{\jmath} + \cos\theta\, \hat{k},$$

$$\hat{\theta} = \cos\varphi \cos\theta\, \hat{\imath} + \sin\varphi \cos\theta\, \hat{\jmath} - \sin\theta\, \hat{k},$$

$$\hat{\varphi} = -\sin\varphi\, \hat{\imath} + \cos\varphi\, \hat{\jmath}.$$

So

$$\hat{r}\cdot\hat{\theta} = \cos^2\varphi \sin\theta\cos\theta + \sin^2\varphi \sin\theta\cos\theta - \sin\theta\cos\theta$$

$$= \sin\theta\cos\theta - \sin\theta\cos\theta = 0,$$

$$\hat{r}\cdot\hat{\varphi} = -\sin\varphi\cos\varphi\sin\theta + \sin\varphi\cos\varphi\sin\theta + 0$$

$$= 0$$

$$\hat{\theta}\cdot\hat{\varphi} = -\sin\varphi\cos\varphi\cos\theta + \sin\varphi\cos\varphi\cos\theta + 0$$

$$= 0.$$

So the coordinate system is orthogonal.

(c) Using the formula on the formula sheet,

$$\vec{\nabla}\theta = \frac{1}{h_r}\frac{\partial f}{\partial r}\hat{r} + \frac{1}{h_\theta}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{h_\varphi}\frac{\partial f}{\partial \varphi}\hat{\varphi}$$

$$= \frac{1}{1}\cdot 0\cdot\hat{r} + \frac{1}{r}\cdot 1\cdot\hat{\theta} + \frac{1}{r\sin\theta}\cdot 0\cdot\hat{\varphi}$$

$$= \frac{1}{r}\hat{\theta}$$

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(d) Using the formula on the formula sheet

$$\vec{\nabla}\cdot(\sin\theta\,\hat{\theta}) = \frac{1}{h_r h_\theta h_\varphi}\left(\frac{\partial(h_\theta h_\varphi F_r)}{\partial r} + \frac{\partial(h_r h_\varphi F_\theta)}{\partial\theta} + \frac{\partial(h_r h_\theta F_\varphi)}{\partial\varphi}\right)$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial(r^2\sin\theta\cdot 0)}{\partial r} + \frac{\partial(1\cdot r\sin\theta\sin\theta)}{\partial\theta} + \frac{\partial(1\cdot r\cdot 0)}{\partial\varphi}\right)$$

$$= \frac{1}{r^2\sin\theta}\left(0 + 2r\sin\theta\cos\theta + 0\right)$$

$$= \frac{2\cos\theta}{r}$$