School of Mathematics and Statistics
MAST20009 Vector Calculus, Semester 2 2018
Assignment 4 and Cover Sheet

Student Name	*Student Number*
Tutor's Name	*Tutorial Day/Time*

Submit your assignment to your tutor's MAST20009 assignment box before 11am on Tuesday 16th October 2017.

This assignment is worth 5% of your final MAST20009 mark.
Please attach this cover sheet to your assignment.

Note:
- Full working must be shown in your solutions.
- Assignments must be neatly handwritten in blue or black pen. Diagrams can be drawn in pencil.
- Marks will be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- There are *four* questions on this assignment.

1. (a) Use

$$\sinh t = \frac{e^t - e^{-t}}{2}, \quad \cosh t = \frac{e^t + e^{-t}}{2}, \quad \tanh t = \frac{\sinh t}{\cosh t}, \quad \text{and} \quad \text{sech } t = \frac{1}{\cosh t}$$

to show that $1 - \tanh^2 t = \text{sech}^2 t$,

$$\frac{d \ \tanh t}{dt} = \text{sech}^2\ t, \qquad \lim_{t\to 0} \tanh t = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \tanh t = 1.$$

(b) Use the change of variables $x = r\cos\theta$, $y = r\sin\theta$, $z = t - \tanh t$ to derive the formula

$$(\text{Volume of } V) = \iiint_V r \tanh^2 t dr\, d\theta\, dt$$

(c) Let S be the parametrised surface (the pseudosphere) given by

$$\Phi(t,\theta) = \left(\frac{\cos\theta}{\cosh t}, \frac{\sin\theta}{\cosh t}, t - \tanh t\right), \qquad \text{for} \quad 0 \le t \text{ and } 0 \le \theta < 2\pi.$$

Let V be the region in $\mathbb{R}^3$ between the plane $z = 0$ and the surface S.

(ca) Graph the region V.

(cb) Compute the volume of the region V.

2. Let C be the directed curve forming the triangle $(0,0,0)$ to $(0,1,1)$ to $(1,1,1)$ to $(0,0,0)$.

(a) Let $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$. Is the field $\mathbf{F}$ conservative? Find $\int_C \mathbf{F}\cdot d\mathbf{s}$.

(b) Let $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$. Is the field $\mathbf{F}$ conservative? Find $\int_C \mathbf{F}\cdot d\mathbf{s}$.

3. Let $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and let S be the flat triangular surface with vertices $(0,0,0), (1,1,0), (2,2,2)$. Assume S is oriented towards the postive x-axis. Find

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

4. Let C be the curve $\mathbf{c}(t) = (t, \sin t)$, for $0 \le t \le \pi$. Use Green's theorem to evaluate the line integral

$$\int_C (3y\,dx + 2x\,dy).$$

(a) Use $\sinh t = \frac{e^t - e^{-t}}{2}$, $\cosh t = \frac{e^t + e^{-t}}{2}$

$\tanh t = \frac{\sinh t}{\cosh t}$ and $\operatorname{sech} t = \frac{1}{\cosh t}$

to show that

$1 - \tanh^2 t = \operatorname{sech}^2 t$, $\frac{d}{dt}\tanh t = \operatorname{sech}^2 t$,

$\lim_{t \to 0} \tanh t = 0$ and $\lim_{t \to \infty} \tanh t = 1$.

Solution:

Since $\cosh^2 t - \sinh^2 t = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2$

$$= \frac{1}{4}(e^{2t} + 2 + e^{-2t}) - \frac{1}{4}(e^{2t} - 2 + e^{-2t})$$

$$= \frac{1}{4}(2+2) = \frac{4}{4} = 1,$$

then

$$1 - \tanh^2 t = 1 - \frac{\sinh^2 t}{\cosh^2 t} = \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t}$$

$$= \frac{1}{\cosh^2 t} = \left(\frac{1}{\cosh t}\right)^2 = \operatorname{sech}^2 t.$$

Since

$$\frac{d \sinh t}{dt} = \frac{d}{dt}\frac{1}{2}(e^t - e^{-t}) = \frac{1}{2}(e^t + e^{-t}) = \cosh t$$

$$\frac{d \cosh t}{dt} = \frac{d}{dt}\frac{1}{2}(e^t + e^{-t}) = \frac{1}{2}(e^t - e^{-t}) = \sinh t$$

then

$$\frac{d}{dt}\tanh t = \frac{d}{dt}\left(\frac{\sinh t}{\cosh t}\right)$$

$$= \sinh t\,(-1)\,\frac{1}{\cosh^2 t}\sinh t + \cosh t\,\frac{1}{\cosh t}$$

$$= -\frac{\sinh^2 t}{\cosh^2 t} + 1 = 1 - \tanh^2 t = \operatorname{sech}^2 t.$$

Next

$$\lim_{t\to 0}\tanh t = \lim_{t\to 0}\frac{\sinh t}{\cosh t} = \lim_{t\to 0}\frac{e^t - e^{-t}}{e^t + e^{-t}}$$

$$= \lim_{t\to 0}\frac{(e^t - e^{-t})}{(e^t + e^{-t})}\cdot\frac{e^t}{e^t} = \lim_{t\to 0}\frac{e^{2t}-1}{e^{2t}+1} = \frac{1-1}{1+1} = 0.$$

and

$$\lim_{t\to\infty}\tanh t = \lim_{t\to\infty}\frac{\sinh t}{\cosh t} = \lim_{t\to\infty}\frac{e^t - e^{-t}}{e^t + e^{-t}}$$

$$= \lim_{t\to\infty}\frac{(e^t - e^{-t})}{(e^t + e^{-t})}\,\frac{e^{-t}}{e^{-t}} = \lim_{t\to\infty}\frac{1-e^{-2t}}{1+e^{-2t}} = \frac{1-0}{1+0}$$

$$= \frac{1}{1} = 1.$$

(b) Use the change of variables

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = t - \tanh t$$

to derive the formula

$$\text{Volume of } V = \iiint_V r\tanh^2 t \, dr\, d\theta\, dt$$

for (appropriately well behaved) regions V in $\mathbb{R}^3$.

Solution

$$\text{Volume of } V = \iiint_V dV = \iiint_V dx\,dy\,dz$$

$$= \iiint_V \left|\det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,t)}\right)\right| dr\,d\theta\,dt.$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,t)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1-\operatorname{sech}^2 t \end{pmatrix}$$

So

$$\det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,t)}\right) = (1-\operatorname{sech}^2 t)\left(r\cos^2\theta - (-r\sin^2\theta)\right)$$

$$= \tanh^2 t\,(r\cos^2\theta + r\sin^2\theta)$$

$$= r\tanh^2 t.$$

So

$$\text{Volume of } V = \iiint_V r\tanh^2 t\, dr\,d\theta\,dt.$$

(c) Let S be the surface parametrized by

$$\Phi(t,\theta) = \left(\frac{\cos\theta}{\cosh t}, \frac{\sin\theta}{\sinh t}, t - \tanh t\right)$$

for $0 \le \theta \le 2\pi$ and $0 \le t \le \infty$.

Let V be the region in $\mathbb{R}^3$ between the plane $z = 0$ and the surface S.

Graph the region V.

<u>Solution:</u> Letting

$$r(t,\theta) = \sqrt{x(t,\theta)^2 + y(t,\theta)^2}$$

$$= \sqrt{\frac{\cos^2\theta}{\cosh^2 t} + \frac{\sin^2\theta}{\cosh^2 t}} = \sqrt{\frac{1}{\cosh^2 t}} = \frac{1}{\cosh t}$$

for points on the surface S.

- When $t = 0$ then $r = \frac{1}{\cosh 0} = \frac{1}{1} = 1$ and

  $$z = 0 - \tanh 0 = 0 - \frac{\sinh 0}{\cosh 0} = 0 - 0 = 0.$$

- As t grows from 0 to ∞ then r goes from 1 to 0 and z goes from 0 to ∞.

So the region V looks like

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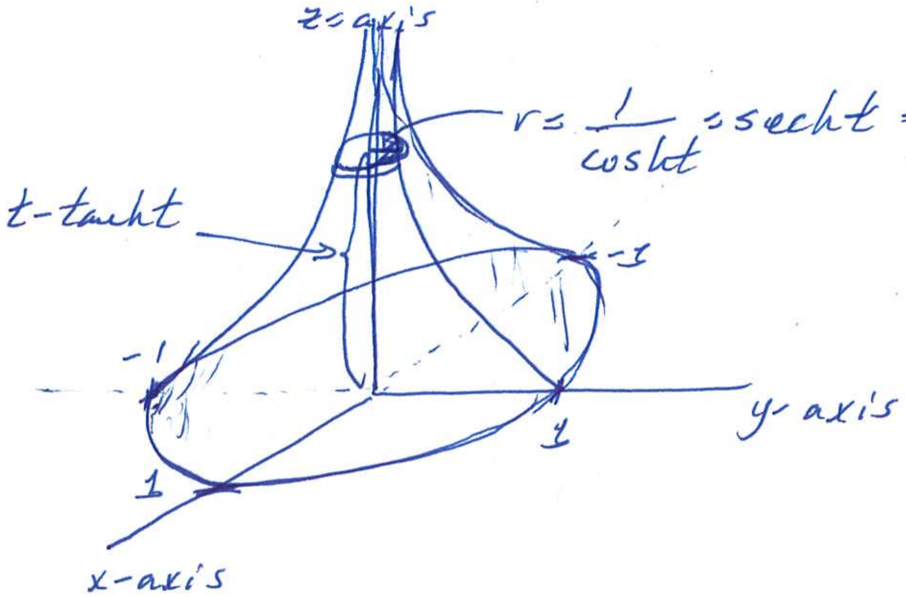
(5)



(d) Compute the volume of V.

Solution Using the formula from part (b)

$$\text{Volume of } V = \iiint_V r \tanh^2 t \, dr \, d\theta \, dt$$

$$= \int_{t=0}^{t=\infty} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\operatorname{sech} t} r \tanh^2 t \, dr \, d\theta \, dt$$

$$= \int_{t=0}^{t=\infty} \int_{\theta=0}^{\theta=2\pi} \tanh^2 t \left. \frac{r^2}{2} \right]_{r=0}^{r=\operatorname{sech} t} d\theta \, dt$$

$$= \int_{t=0}^{t=\infty} \int_{\theta=0}^{\theta=2\pi} \tanh^2 t \, \frac{\operatorname{sech}^2 t}{2} \, d\theta \, dt$$

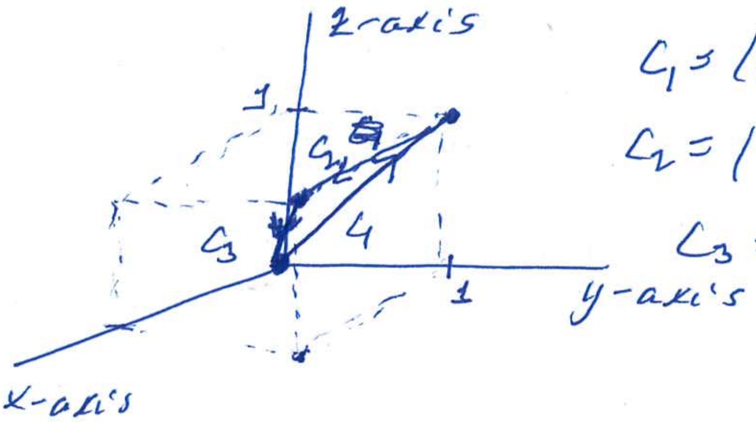
$$= \int_{t=0}^{t=\infty} \tanh^2 t \, \frac{\operatorname{sech}^2 t}{2} \left. \theta \right]_{\theta=0}^{\theta=2\pi} dt$$

$$= \int_{t=0}^{t=\infty} \tanh^2 t \, \frac{\operatorname{sech}^2 t}{2} \, 2\pi \, dt$$

$$= \pi \left. \frac{\tanh^3 t}{3} \right]_{t=0}^{t=\infty} = \frac{\pi}{3}(1-0) = \frac{\pi}{3}.$$

(2) Let C be the directed curve framing the triangle
$(0,0,0)$ to $(0,1,1)$ to $(1,1,1)$ to $(0,0,0)$

(a) Let $\vec{F} = x\hat{i} + xy\hat{j} + xz\hat{k}$.



$C_1 = (0, t, t) \quad 0 \leq t \leq 1$

$C_2 = (t, 1, 1) \quad 0 \leq t \leq 1$

$C_3 = (1-t, 1-t, 1-t) \quad 0 \leq t \leq 1.$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & xy & xz \end{vmatrix} = \begin{matrix} \hat{i}(0-0) \\ -\hat{j}(z-0) \\ +\hat{k}(y-0) \end{matrix} = \begin{matrix} -z\hat{j} \\ +y\hat{k} \end{matrix} \neq \vec{0}$$

So $\vec{F}$ is not conservative.

$$\int_C \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} + \int_{C_3} \vec{F} \cdot d\vec{s}$$

$$= \int_{C_1} (x\hat{i} + yx\hat{j} + xz\hat{k}) \cdot (0\hat{i} + 1\hat{j} + 1\hat{k})\, dt$$

$$+ \int_{C_2} (x\hat{i} + xy\hat{j} + xz\hat{k}) \cdot (\hat{i} + 0\hat{j} + 0\hat{k})\, dt$$

$$+ \int_{C_3} (x\hat{i} + xy\hat{j} + xz\hat{k}) \cdot (-\hat{i} - \hat{j} - \hat{k})\, dt$$

$$= \int_{t=0}^{t=1} (0\hat{\imath} + 0\cdot t\,\hat{\jmath} + 0\cdot t\,\hat{k})\cdot(0\hat{\imath} + \hat{\jmath} + \hat{k})\,dt$$

$$+ \int_{t=0}^{t=1} (t\hat{\imath} + t\cdot 1\,\hat{\jmath} + t\cdot 1\,\hat{k})\cdot(\hat{\imath} + 0\hat{\jmath} + 0\hat{k})\,dt$$

$$+ \int_{t=0}^{t=1} \left((1-t)\hat{\imath} + (1-t)^2\hat{\jmath} + (1-t)^2\hat{k}\right)\cdot(-\hat{\imath} - \hat{\jmath} - \hat{k})\,dt$$

$$= 0 + \int_{t=0}^{t=1} t\,dt + \int_{t=0}^{t=1} \left(-(1-t) - (1-t)^2 - (1-t)^2\right)dt$$

$$= \frac{t^2}{2}\Bigg]_{t=0}^{t=1} + \int_{t=0}^{t=1} \left(-1+t-2(1-2t+t^2)\right)dt$$

$$= \frac{1}{2} - 0 + \int_{t=0}^{t=1} (-1+t-2+4t-2t^2)\,dt$$

$$= \frac{1}{2} + \int_{t=0}^{t=1} (-2t^2+5t-3)\,dt$$

$$= \frac{1}{2} + \left(-2\frac{t^3}{3} + 5\frac{t^2}{2} - 3t\right)\Bigg]_{t=0}^{t=1}$$

$$= \frac{1}{2} + \left(-\frac{2}{3} + \frac{5}{2} - 3\right) - (0+0-0)$$

$$= \frac{6}{2} - \frac{2}{3} - 3 = 3 - \frac{2}{3} - 3 = -\frac{2}{3}.$$

(b) Let $\vec{F} = yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \begin{matrix} \hat{\imath}(x-x) \\ -\hat{\jmath}(y-y) \\ +\hat{k}(z-z) \end{matrix} = \vec{0}.$$

So $\vec{F}$ is conservative.

Guess for f so that $\vec{F} = \vec{\nabla} f$: Try $f = xyz$.

Check: $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial(xyz)}{\partial x}\hat{\imath} + \frac{\partial(xyz)}{\partial y}\hat{\jmath} + \frac{\partial(xyz)}{\partial z}\hat{k}$$

$$= yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}$$

$$= \vec{F}.$$

So $\vec{F} = \vec{\nabla} f$ with $f = xyz$.

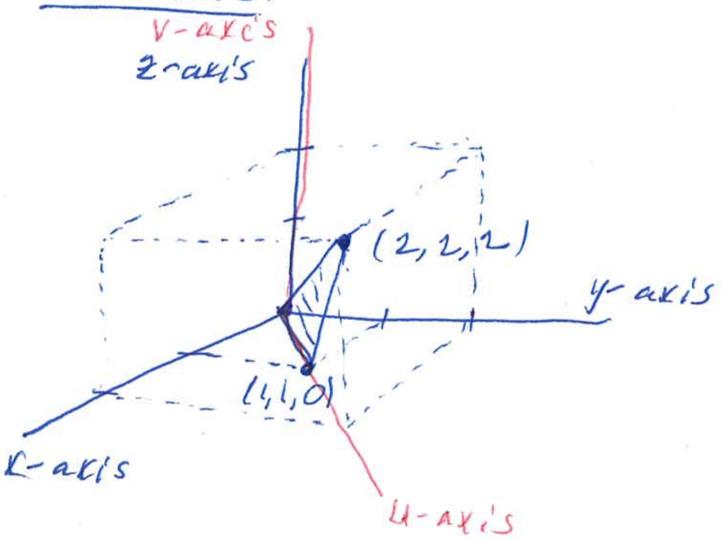
So
$$\int_C \vec{F}\cdot d\vec{s} = f\begin{pmatrix}\text{ending point}\\ \text{on } C\end{pmatrix} - f\begin{pmatrix}\text{initial point}\\ \text{on } C\end{pmatrix}$$

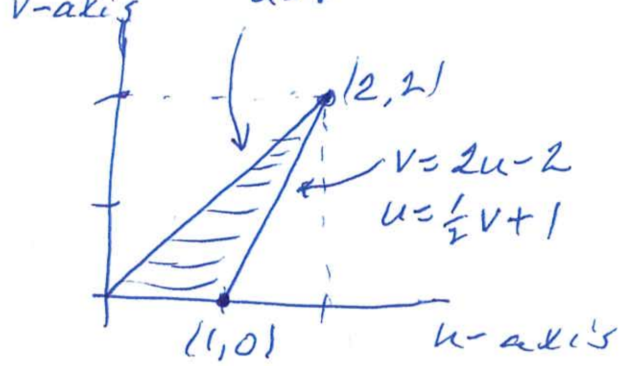
$$= f(0,0,0) - f(0,0,0) = 0.$$

3) Let $\vec{F} = x^2\hat{\imath} + xy\hat{\jmath} + z\hat{k}$ and let S be the flat triangular surface with vertices $(0,0,0)$, $(1,1,0)$ and $(2,2,2)$. Find

$$\iint_S \vec{F}\cdot d\vec{S}.$$

Solution:





The triangular surface S has $x=y$ at all points and is parametrised by

$S = (u,u,v)$ with $v \le u \le \frac{1}{2}v+1$ and $0 \le v \le 2$.

Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot\hat{n}\,|\vec{T}_u \times \vec{T}_v|\,du\,dv$$

$$= \iint_S \vec{F}\cdot\left(\frac{\vec{T}_u \times \vec{T}_v}{|\vec{T}_u \times \vec{T}_v|}\right)|\vec{T}_u \times \vec{T}_v|\,du\,dv$$

$$= \iint_S \vec{F} \cdot (\vec{T}_u \times \vec{T}_v)\, du\, dv$$

$$= \int_{v=0}^{v=2} \int_{u=v}^{u=\frac{1}{2}v+1} \vec{F} \cdot (\vec{T}_u \times \vec{T}_v)\, du\, dv$$

Since S is parametrized by $\Phi = (u, u, v)$ then

$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (1, 1, 0)$ and

$\vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (0, 0, 1)$ and

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k} \cdot 0 = \hat{i} - \hat{j} + 0\hat{k}.$$

So $$\iint_S \vec{F} \cdot d\vec{S} = \int_{v=0}^{v=2} \int_{u=v}^{u=\frac{1}{2}v+1} \vec{F} \cdot (\vec{T}_u \times \vec{T}_v)\, du\, dv$$

$$= \int_{v=0}^{v=2} \int_{u=v}^{u=\frac{1}{2}v+1} (x^2\hat{i} + xy\hat{j} + z\hat{k}) \cdot (\hat{i} - \hat{j} + 0\hat{k})\, du\, dv$$

$$= \int_{v=0}^{v=2} \int_{u=v}^{u=\frac{1}{2}v+1} (u^2\hat{i} + u \cdot u\hat{j} + v\hat{k}) \cdot (\hat{i} - \hat{j} + 0\hat{k})\, du\, dv$$

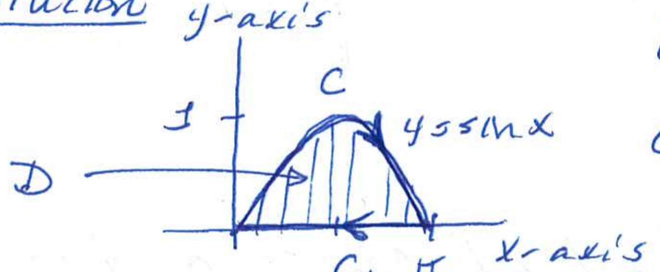
$$= \int_{v=0}^{v=2} \int_{u=v}^{u=\frac{1}{2}v+1} (u^2 - u^2 + 0)\, du\, dv$$

$$= \int_{v=0}^{v=2} \int_{u=v}^{u=\frac{1}{2}v+1} 0\, du\, dv = 0.$$

(4) Let $C(t) = (t, \sin t)$ with $0 \le t \le \pi$.
Use Green's theorem to evaluate the line integral

$$\int_C 3y\,dx + 2x\,dy.$$

<u>Solution</u>



∂D is $-(C \cup C_2)$ where $C_2(t) = (\pi - t, 0)$ with $0 \le t \le \pi$.

Green's theorem says

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

In our case $P = 3y$ and $Q = 2x$, so

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{x=0}^{x=\pi} \int_{y=0}^{y=\sin x} \left(\frac{\partial (2x)}{\partial x} - \frac{\partial (3y)}{\partial y}\right) dy\,dx$$

$$= \int_{x=0}^{x=\pi} \int_{y=0}^{y=\sin x} (2-3)\,dy\,dx = \int_{x=0}^{x=\pi} -y\Big]_{y=0}^{y=\sin x} dx$$

$$= \int_{x=0}^{x=\pi} (-\sin x - 0)\,dx = \cos x\Big]_{x=0}^{x=\pi} = \cos\pi - \cos 0$$

$$= (-1) - 1 = -2.$$

So $-2 = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$= \int_{\partial D} P\,dx + Q\,dy$

$= \int_{-(C \cup C_2)} 3y\,dx + 2x\,dy$

$= -\left(\int_C 3y\,dx + 2x\,dy\right) + \left(\int_{C_2} 3y\,dx + 2x\,dy\right)$

$= -\left(\int_C 3y\,dx + 2x\,dy\right) + \int_{C_2} \left(3y\frac{dx}{dt} + 2x\frac{dy}{dt}\right) dt$

$= -\left(\int_C 3y\,dx + 2x\,dy\right) + \int_{t=0}^{t=\pi} \left(3 \cdot 0 \cdot \frac{d(\pi - t)}{dt} + 2(\pi - t)\frac{d0}{dt}\right) dt$

$= -\left(\int_C 3y\,dx + 2x\,dy\right) + \int_{t=0}^{t=\pi} (0 + 0)\,dt$

$= -\left(\int_C 3y\,dx + 2x\,dy\right) + 0.$

So

$\int_C 3y\,dx + 2x\,dy = +2.$