School of Mathematics and Statistics

MAST20009 Vector Calculus, Semester 2 2018

Assignment 3 and Cover Sheet

Student Name	*Student Number*
Tutor's Name	*Tutorial Day/Time*

Submit your assignment to your tutor's MAST20009 assignment box before 11am on Tuesday 2nd October.

This assignment is worth 5% of your final MAST20009 mark.
Please attach this cover sheet to your assignment.

Note:
- Full working must be shown in your solutions.
- Assignments must be neatly handwritten in blue or black pen. Diagrams can be drawn in pencil.
- Marks will be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- There are 3 questions on this assignment.

1. Evaluate the double integral
$$\iint_R \sqrt{(x+y)(x-2y)}\,dx\,dy$$
where R is the region enclosed between $y = \frac{1}{2}x$, $y = 0$ and $x + y = 1$.
(*Hint*: Use new coordinates $u = x + y$, $v = x - 2y$.)

2. (a) Use
$$\sinh t = \frac{e^t - e^{-t}}{2} \qquad \text{and} \qquad \cosh t = \frac{e^t + e^{-t}}{2}$$
to show that $\cosh^2 t - \sinh^2 t = 1$,
$$\frac{d\ \sinh t}{dt} = \cosh t, \quad \frac{d\ \cosh t}{dt} = \sinh t, \quad \lim_{t\to 0}\frac{1}{\cosh t} = 1, \quad \text{and} \quad \lim_{t\to\infty}\frac{1}{\cosh t} = 0,$$
   (b) Let S be the parametrised surface (the pseudosphere) given by
$$\Phi(t,\theta) = \left(\frac{\cos\theta}{\cosh t}, \frac{\sin\theta}{\cosh t}, t - \tanh t\right), \qquad \text{for} \quad 0 \le t \text{ and } 0 \le \theta < 2\pi.$$
Calculate the surface area of S.

3. Let R be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$ and $z = x + y$.
   (a) Find the volume of R.
   (b) Assume that the region is a solid of uniform density and use triple integration to find its center of mass.

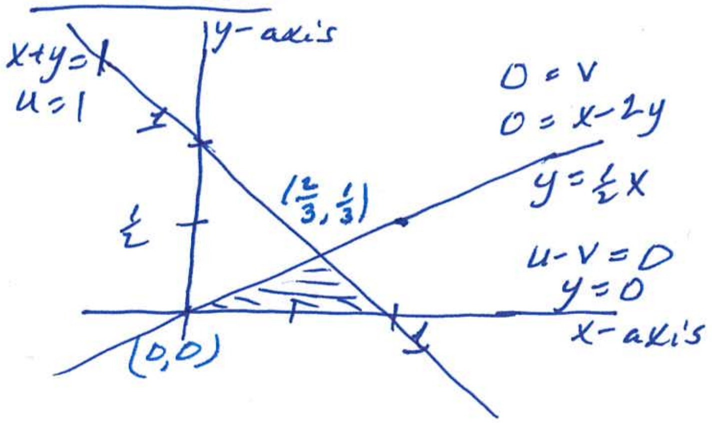
(1) Compute $\iint_R \sqrt{(x+y)(x-2y)}\, dx\,dy$, where

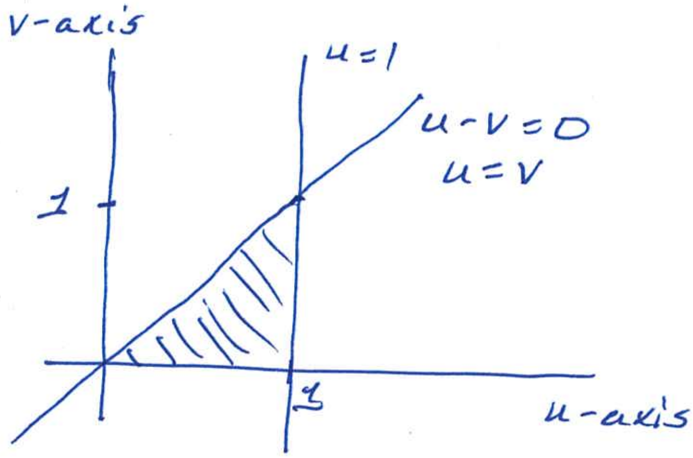
R is the region bounded by

$y = \frac{1}{2}x, \quad y = 0$ and $x + y = 1$.

Hint: $u = x+y$ and $v = x - 2y$.

Solution:





$u = x+y$, so $u - v = 3y$, so $x = \frac{2}{3}u + \frac{1}{3}v$

$v = x - 2y$. $\quad v + 2u = 3x$. $\quad y = \frac{1}{3}u - \frac{1}{3}v$.

So $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{pmatrix}$ so $\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right) = \frac{-2}{9} - \frac{1}{9} = \frac{-3}{9}$

Then

$$\iint_R \sqrt{(x+y)(x-2y)}\, dx\,dy = \iint_R \sqrt{uv}\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| dv\,du$$

$$= \iint_R u^{\frac{1}{2}} v^{\frac{1}{2}} \left|\frac{-3}{9}\right| dv\,du = \int_{u=0}^{u=1}\int_{v=0}^{v=u} \frac{1}{3} u^{\frac{1}{2}} v^{\frac{1}{2}}\, dv\,du$$

Ass 3 Q 1

(2)

$$= \int_{u=0}^{u=1} \frac{2}{3}\cdot\frac{1}{3} u^{\frac{1}{2}} v^{3/2} \Big]_{v=0}^{v=u} du$$

$$= \int_{u=0}^{u=1} \frac{2}{9}(u^2 - 0)\, du = \frac{2}{9}\frac{u^3}{3}\Big]_{u=0}^{u=1}$$

$$= \frac{2}{27}(1-0) = \frac{2}{27}.$$

(a) Use $\sinh t = \frac{e^t - e^{-t}}{2}$ and $\cosh t = \frac{e^t + e^{-t}}{2}$

to show that $\cosh^2 t - \sinh^2 t = 1$

$\frac{d \sinh t}{dt} = \cosh t$, $\frac{d \cosh t}{dt} = \sinh t$,

$\lim_{t \to 0} \frac{1}{\cosh t} = 1$ and $\lim_{t \to \infty} \frac{1}{\cosh t} = 0$.

Solution:

(aa) $\cosh^2 t - \sinh^2 t = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2$

$$= \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = \frac{2+2}{4} = 1.$$

(ab) $\frac{d \sinh t}{dt} = \frac{d}{dt} \frac{1}{2}(e^t - e^{-t}) = \frac{1}{2}(e^t - (-1)e^{-t})$

$$= \frac{1}{2}(e^t + e^{-t}) = \cosh t.$$

(ac) $\frac{d \cosh t}{dt} = \frac{d}{dt} \frac{1}{2}(e^t + e^{-t}) = \frac{1}{2}(e^t + (-1)e^{-t})$

$$= \frac{1}{2}(e^t - e^{-t}) = \sinh t$$

(ad) $\lim_{t \to 0} \frac{1}{\cosh t} = \lim_{t \to 0} \frac{2}{e^t + e^{-t}}$ $\cancel{= \lim_{t \to 0} \frac{2}{e^t + 1} =}$

$$= \frac{2}{1+1} = \frac{2}{2} = 1.$$

Since $e^t \in \mathbb{R}_{>0}$ for $t \in \mathbb{R}$ then Ass 3 Q 2

(ae) $0 \le \frac{2}{e^t + e^{-t}} \le \frac{2}{e^t + 0} = \frac{2}{e^t}$ for $t \in \mathbb{R}$ (2)

So

$$\lim_{t \to \infty} 0 \le \lim_{t \to \infty} \frac{2}{e^t + e^{-t}} \le \lim_{t \to \infty} \frac{2}{e^t}$$

So

$$0 \le \lim_{t \to \infty} \frac{1}{\cosh t} \le 0.$$

So $\lim_{t \to \infty} \frac{1}{\cosh t} = 0.$

(b) The parametrised surface S is

$$\Phi(t,\theta) = \left(\frac{\cos\theta}{\cosh t}, \frac{\sin\theta}{\cosh t}, t - \tanh t\right)$$

for $0 \le \theta \le 2\pi$ and $0 \le t \le \infty$. Then

$$\text{Surface area of } S = \iint_S |\vec{T}_t \times \vec{T}_\theta| \, dt \, d\theta$$

Then
$$\vec{T}_t = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) \text{ and}$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\cos\theta}{\cosh t}\right) = \cos\theta\left(\frac{-1}{\cosh^2 t}\right)\sinh t = \frac{-\cos\theta\sinh t}{\cosh^2 t}$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\sin\theta}{\cosh t}\right) = \sin\theta\left(\frac{-1}{\cosh^2 t}\right)\sinh t = \frac{-\sin\theta\sinh t}{\cosh^2 t}$$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}(t - \tanh t) = \frac{\partial}{\partial t}\left(t - \frac{\sinh t}{\cosh t}\right)$$

$$= 1 - \left(\sinh t\left(\frac{-1}{\cosh^2 t}\right)\sinh t + \cosh t \frac{1}{\cosh t}\right)$$

$$= 1 - \left(\frac{-\sinh^2 t}{\cosh^2 t} + 1\right) = \cancel{1 - \left(\frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t}\right)}$$

$$= \frac{\sinh^2 t}{\cosh^2 t}$$

So $\vec{r}_t = \left(-\frac{\cos\theta \sinh t}{\cosh^2 t}, \frac{-\sin\theta \sinh t}{\cosh^2 t}, \frac{\sinh^2 t}{\cosh^2 t} \right)$

$$= \frac{\sinh t}{\cosh^2 t}(-\cos\theta, -\sin\theta, \sinh t)$$

Next,

$$\vec{r}_\theta = \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right) = \left(\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\cosh t}\right), \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh t}\right), \frac{\partial (t - \tanh t)}{\partial\theta} \right)$$

$$= \left(\frac{1}{\cosh t}(-\sin\theta), \frac{1}{\cosh t}(\cos\theta), 0 \right)$$

$$= \frac{1}{\cosh t}(-\sin\theta, \cos\theta, 0)$$

So $\vec{r}_t \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\sinh t}{\cosh^2 t}(-\cos\theta) & \frac{\sinh t}{\cosh t}(-\sin\theta) & \frac{\sinh t}{\cosh t}\sinh t \\ \frac{1}{\cosh t}(-\sin\theta) & \frac{1}{\cosh t}\cos\theta & 0 \end{vmatrix}$

$$= \frac{\sinh t}{\cosh^2 t}\frac{1}{\cosh t} \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\cos\theta & -\sin\theta & \sinh t \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= \frac{\sinh t}{\cosh^3 t}\left(\hat{i}(-\cos\theta \sinh t) + \hat{k}(-\cos^2\theta - \sin^2\theta) - \hat{j}(\sin\theta \sinh t) \right)$$

$$= \frac{\sinh t}{\cosh^3 t}(-\cos\theta \sinh t, -\sin\theta \sinh t, -1)$$

So $\left|\vec{T}_t \times \vec{T}_\theta\right| = \frac{\sinh t}{\cosh^3 t} \sqrt{\cos^2\theta \sinh^2 t + \sin^2\theta \sinh^2 t + (-1)^2}$

$$= \frac{\sinh t}{\cosh^3 t} \sqrt{\sinh^2 t + 1} = \frac{\sinh t}{\cosh^3 t} \sqrt{\cosh^2 t}$$

$$= \frac{\sinh t}{\cosh^2 t}.$$

So surface area of S $= \iint_S \left|\vec{T}_t \times \vec{T}_\theta\right| \, dt \, d\theta$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{t=0}^{t=\infty} \frac{1}{\cosh^2 t} \sinh t \, dt \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{-1}{\cosh t} \Big]_{t=0}^{t=\infty} d\theta$$

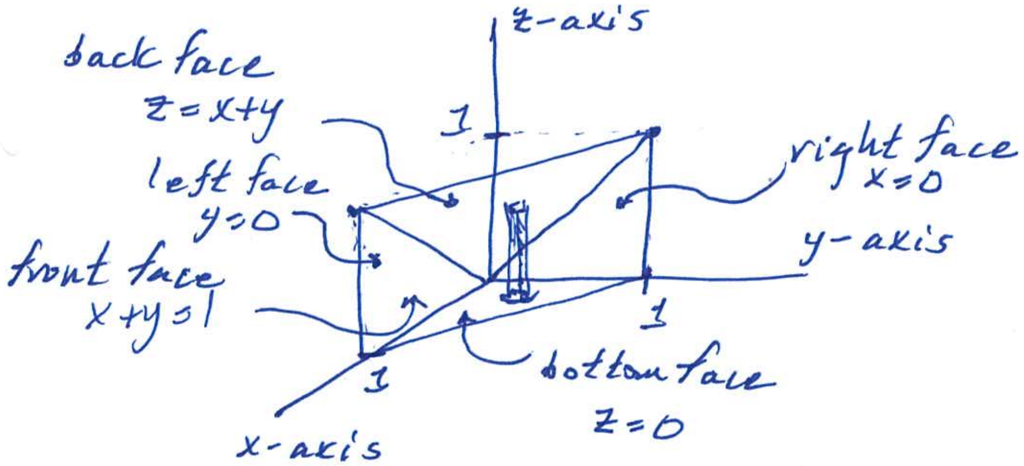
$$= \int_{\theta=0}^{\theta=2\pi} \left(-0 - \frac{-1}{1}\right) d\theta = \int_{\theta=0}^{\theta=2\pi} d\theta$$

$$= \theta \Big]_{\theta=0}^{\theta=2\pi} = 2\pi.$$

(3) Let R be the region bounded by

$x=0,\ y=0,\ z=0,\ x+y=1,\ z=x+y.$

(a) Compute the volume of R.



$$\text{Volume} = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} \int_{z=0}^{z=x+y} dz\, dx\, dy$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} z \Big]_{z=0}^{z=x+y} dx\, dy$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} (x+y)\, dx\, dy$$

$$= \int_{y=0}^{y=1} \frac{x^2}{2} + yx \Big]_{x=0}^{x=1-y} dy$$

$$= \int_{y=0}^{y=1} \left(\frac{(1-y)^2}{2} + y(1-y)\right) dy$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{2}(1-2y+y^2) + y - y^2\right) dy$$

$$= \int_{y=0}^{y=1} \left(\tfrac{1}{2} - \tfrac{1}{2}y^2\right) dy = \tfrac{1}{2}\int_{y=0}^{y=1} (1-y^2)\, dy$$

$$= \tfrac{1}{2}\left(y - \tfrac{y^3}{3}\right)\Big]_{y=0}^{y=1} = \tfrac{1}{2}\left(1 - \tfrac{1}{3} - 0 + 0\right)$$

$$= \tfrac{1}{2} \cdot \tfrac{2}{3} = \tfrac{1}{3}.$$

(b) Find the centre of mass of R (assuming uniform density).

$$\begin{array}{l}\text{x-coordinate} \\ \text{of centre of mass}\end{array} = \frac{\int_{y=0}^{y=1}\int_{x=0}^{x=1-y}\int_{z=0}^{z=x+y} x\, dz\, dx\, dy}{\text{Volume}}$$

$$= \frac{1}{\frac{1}{3}} \int_{y=0}^{y=1}\int_{x=0}^{x=1-y} xz\Big]_{z=0}^{z=x+y}\, dx\, dy$$

$$= 3\int_{y=0}^{y=1}\int_{x=0}^{x=1-y} x(x+y)\, dx\, dy$$

$$= 3\int_{y=0}^{y=1}\left(\frac{x^3}{3} + y\frac{x^2}{2}\Big]_{x=0}^{x=1-y}\right) dy$$

$$= 3\int_{y=0}^{y=1}\left(\tfrac{1}{3}(1-y)^3 + \tfrac{1}{2}y(1-y)^2 - (0+0)\right) dy$$

$$= 3\int_{y=0}^{y=1}\left(\tfrac{1}{3}(1-3y+3y^2-y^3) + \tfrac{1}{2}y(1-2y+y^2)\right) dy$$

$$= 3\int_{y=0}^{y=1}\left(\tfrac{1}{3} - \tfrac{1}{2}y + 0y^2 + \tfrac{1}{6}y^3\right) dy$$

$$= 3\left(\tfrac{1}{3}y - \tfrac{1}{2}\tfrac{y^2}{2} + \tfrac{1}{6}\tfrac{y^4}{4}\right)\Big]_{y=0}^{y=1}$$

$$= 3\left(\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{6\cdot 4} - 0\right) = 1 - \tfrac{3}{4} + \tfrac{1}{2\cdot 4} = \tfrac{1}{4} + \tfrac{1}{8}$$

$$= \tfrac{3}{8}.$$

z-coordinate of centre of mass $= \dfrac{\int_{y=0}^{y=1}\int_{x=0}^{x=1-y}\int_{z=0}^{z=x+y} z\, dz\, dx\, dy}{\text{Volume}}$

$$= 3\int_{y=0}^{y=1}\int_{x=0}^{x=1-y} \frac{z^2}{2}\Big]_{z=0}^{z=x+y} dx\, dy$$

$$= 3\int_{y=0}^{y=1}\int_{x=0}^{x=1-y} \frac{(x+y)^2}{2}\, dx\, dy$$

$$= 3\int_{y=0}^{y=1} \frac{1}{2}\frac{(x+y)^3}{3}\Big]_{x=0}^{x=1-y} dy$$

$$= \frac{1}{2}\int_{y=0}^{y=1}(1-y^3)\, dy = \frac{1}{2}\left(y - \frac{y^4}{4}\right)\Big]_{y=0}^{y=1}$$

$$= \frac{1}{2}\left(1 - \frac{1}{4} - (0-0)\right) = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}.$$

One might conclude, by symmetry, that

$$\begin{array}{c}\text{x-coordinate}\\ \text{of centre of mass}\end{array} = \begin{array}{c}\text{y-coordinate}\\ \text{of centre of mass}\end{array}$$

or complete the computation as follows:

$$\begin{array}{c}\text{y-coordinate}\\ \text{of centre of mass}\end{array} = \frac{\int_{y=0}^{y=1}\int_{x=0}^{x=1-y}\int_{z=0}^{z=x+y} y\,dz\,dx\,dy}{\text{Volume}}$$

$$= 3\int_{y=0}^{y=1}\int_{x=0}^{x=1-y} yz\Big]_{z=0}^{z=x+y} dx\,dy$$

$$= 3\int_{y=0}^{y=1}\int_{x=0}^{x=1-y} y(x+y)\,dx\,dy$$

$$= 3\int_{y=0}^{y=1} y\frac{(x+y)^2}{2}\Big]_{x=0}^{x=1-y} dy$$

$$= 3\int_{y=0}^{y=1} \frac{y}{2}\left(1^2-y^2\right)dy = \frac{3}{2}\int_{y=0}^{y=1}(y-y^3)\,dy$$

$$= \frac{3}{2}\left(\frac{y^2}{2}-\frac{y^4}{4}\right)\Big]_{y=0}^{y=1} = \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{3}{2}\cdot\frac{1}{4}$$

$$= \frac{3}{8}.$$