Vector Calculus Semester 2 Exams and Assignments 2017 and 2018

Arun Ram   email: aram@unimelb.edu.au

January 8, 2025

Abstract

This is a compendium of exam and assignment questions used for Arun Ram's lectures of Vector Calculus in Semester 2 of the years 2017 and 2018.

Key words— Calculus

Contents

1 Introduction

AMS Subject Classifications: Primary 05E05??; Secondary 33D52??.

2 Final Exam Semester 2 Year 2017

This is a 3 hour exam. No books, notes or calculators are allowed but a formula sheet was provided.

Problem 1. (15 marks) Prove that if $f(x, y)$ is defined for all x, y, by

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$$

then

(a) for any fixed x, $f(x, y)$ is a continuous function of y,

(b) $f(x, y)$ is not continuous at $(0, 0)$,

(c) $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ exist at $(0, 0)$ but are not continuous there.

Problem 2. (8 marks) Using Lagrange Multipliers, find all the extremal points and the extremum values of the function

$$f(x, y) = xy$$

subject to the constraint

$$\tfrac{1}{8}x^2 + \tfrac{1}{2}y^2 - 1 = 0.$$

Justify that the points you have found give the maximum and minimum of f.

Problem 3. (10 marks) Consider the curve

$$\mathbf{c}(t) = \big(2(-t + \sin t), \sqrt{2}(1 - \cos t), \sqrt{2}(1 - \cos t)\big), \qquad \text{with } 0 < t < 2\pi.$$

(a) Find the length of the arc $\mathbf{c}$.

(b) Prove that the unit tangent vector is given by

$$\mathbf{T}(t) = \left(-\sin(t/2), \frac{\sqrt{2}\cos(t/2)}{2}, \frac{\cos(t/2)}{\sqrt{2}}\right),$$

and therefore find the principal normal $\mathbf{N}(t)$ and the binormal $\mathbf{B}(t)$, for $0 < t < 2\pi$.

Problem 4. (10 marks) Let f, g and h be scalar functions of order C^2 on $\mathbb{R}^3$.

(a) Show that $\nabla(e^f) = e^f \nabla f$.

(b) Show that $\nabla^2(e^f) = e^f(\nabla^2 f + \nabla f \cdot \nabla f)$.

(c) Assume that g and h are such that $\nabla^2 g = 0$ and $\nabla^2 h = 0$ and $\nabla g \cdot \nabla h = 0$. Obtain an expression for $\nabla^2(ge^h - he^g)$ in the simplest possible form.

Problem 5. (12 marks) Consider the smaller part of the sphere $x^2 + y^2 + z^2 = 4$ cut by the plane $z = \sqrt{2}$.

(a) Find the volume.

(b) Find the surface area of this portion of the sphere.

Problem 6. (10 marks) Let D be the region bounded by the curves $x = y^2$ and $x = 2y - y^2$. Sketch D and find the moment of intertia about the x-axis of the region D if the density is $\rho = y + 1$.

Problem 7. (10 marks) Let C be the portion of the curve in which the plane $z = 2x + 3y$ cuts the cylinder $x^2 + y^2 = 12$ with $y \geq 0$. Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + (3xy^2 z^2 + 2)\mathbf{k}.$$

(a) Find f such that $\mathbf{F} = \nabla f$.

(b) Calculate the work done by the vector field $\mathbf{F}$ along the curve C (counterclockwise as viewed from the positive end of the z-axis looking toward the origin).

Problem 8. (15 marks) Let S be the closed surface that consists of the

upper hemisphere of radius 1, centred at $(0, 0, 0)$, with base $x^2 + y^2 \leq 1$, $z = 0$.

Let the vector field $\mathbf{F}$ be defined by

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

Verify Gauss' Divergence theorem for the vector field $\mathbf{F}$ and the closed surface S.

Problem 9. (17 marks) Let S be that part of the cube centred at $(0, 0, 1)$ with side length $2\sqrt{2}$ that lies above the x-y plane. Let $\mathbf{F}(x, y, z) = \cos(xz)\mathbf{i} + x^3\mathbf{j} + ye^{-xz}\mathbf{k}$. Evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

using

1. an appropriate line integral;
2. the simplest surface.

Fully justify your answers.

Problem 10. (15 marks)

(a) State Green's theorem and explain all the symbols used and all the required conditions.

(b) Using Green's theorem evaluate the line integral

$$\int_C y^2 dx + x^2 dy \ ,$$

where C is the triangle bounded by $x = 0$, $y = 0$, and $2x + y = 1$ traversed clockwise.

Problem 11. (15 marks) Define *parabolic cylindrical* coordinates (u, v, z) by

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z$$

where $u \in \mathbb{R}, \ v \geq 0, \ z \in \mathbb{R}$ and $u^2 + v^2 > 0$.

1. Compute the scale factors h_u, h_v and h_z.
2. Show that the coordinate system is orthogonal.
3. Write down an expression for the absolute value of the Jacobian.
4. Let $f(u, v, z) = u^3 v^5 + 5678z^2 + 3456$ and $\mathbf{F}(u, v, z) = u^2 \mathbf{e}_u$. Find expressions for

$$\nabla f \qquad \text{and} \qquad \nabla \times \mathbf{F}$$

in terms of u, v and z and $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_z$.

3 Final Exam Makeup Semester 2 Year 2017

Problem 1. (10 marks) Consider the function f given by

$$f(x,y) = \begin{cases} \dfrac{3y^2 - x^3}{x^2 + 2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Calculate $\dfrac{\partial f}{\partial y}$ if $(x,y) \neq (0,0)$.

(b) Calculate $\dfrac{\partial f}{\partial y}$ if $(x,y) = (0,0)$, if it exists.

(c) Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists.

(d) Is f continuous at $(0,0)$? Explain briefly.

Problem 2. (10 marks) Consider the two functions $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\begin{aligned} f(x,y) &= \big(x^2,\ 2y,\ x-y\big) \\ g(u,v,w) &= (u+w,\ v^2+w^2). \end{aligned}$$

(a) Evaluate the derivative $\boldsymbol{D}(f \circ g)$ of the composite function $f \circ g$ at $(x,y,z) = (0,2,-1)$ using the *matrix version of the chain rule.*

(b) Write down an expression for $h(u,v,w) = (f \circ g)(u,v,w)$.

(c) Using part (b), evaluate $\boldsymbol{D}(h)$, and check your answer to part (a) by direct substitution.

Problem 3. (12 marks) A rectangular box having no top and a prescribed volume of $60\,m^3$ is to be constructed using two different materials. The material used for the bottom and the front of the box is 5 times as costly (per m^2) as the material used for the back and the other two sides.
Using the method of *Lagrange Multipliers*, find the dimensions of the box required to minimise the cost of the materials.
Justify that the dimensions found correspond to the minimum cost, and calculate the minimum cost.

Problem 4. (8 marks)

(a) Find the two constants $A, B \in \mathbb{R}$ so that the vector field

$$\boldsymbol{V}(x,y,z) = Ax\sin(\pi y)\boldsymbol{i} + \big(x^2\cos(\pi y) + Bye^{-z}\big)\,\boldsymbol{j} + y^2e^{-z}\boldsymbol{k}$$

is an irrotational vector field.

(b) Using the two constants found in part(a), find a scalar potential function ϕ for $\boldsymbol{V}$.

Problem 5. (10 marks)

(a) (i) If $\boldsymbol{r} = (x, y, z)$, $r = |\boldsymbol{r}|$, $r \neq 0$, and $f(r)$ is a C^1 scalar function show that

$$\boldsymbol{\nabla} \cdot (f(r)\boldsymbol{r}) = r\frac{df}{dr} + 3f(r).$$

(ii) Using part (i), compute $\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{r}}{r^4}\right)$.

(b) Prove vector identity 11, that is, for a C^2 function f, show that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{0}$.

Problem 6. (10 marks)

(a) Find the length of the curve $\boldsymbol{c}(t) = \big(\sin(2t), \cos(2t), 2t^{3/2}\big)$ for $0 \leq t \leq 2$.

(b) Evaluate the integral

$$\int_0^9 \int_{\sqrt{y}}^3 \exp(x^3)\, dx\, dy.$$

Problem 7. (10 marks) Let R be the solid region bounded by the cylinder $x^2 + y^2 = 4$, the plane $z = 0$, and the cone $z = \sqrt{x^2 + y^2}$.

(a) Draw a sketch of the solid region R.

(b) Evaluate the integral

$$\iiint_R (x^2 + y^2)^{3/2}\, dV.$$

Problem 8. (10 marks) Let S be the surface of the upper hemisphere $x^2 + y^2 + z^2 = 16$ for $z \geq 0$.

(a) By parameterizing S, find a normal vector to S.

(b) Hence evaluate

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S},$$

where $\boldsymbol{F}(x, y, z) = -y\boldsymbol{i} + x\boldsymbol{j} - \boldsymbol{k}$, using the outward normal to S.

Problem 9. (10 marks) Let S be the capped cylindrical surface given by the union of two surfaces S_1 and S_2 where S_1 is $x^2 + y^2 = 1$, with $-2 \leq z \leq 0$, and S_2 is $z = 1 - x^2 - y^2$ with $0 \leq z \leq 1$. Let $\boldsymbol{F}$ be the vector field given by

$$\boldsymbol{F}(x, y, z) = yz\boldsymbol{i} + xyz^2\boldsymbol{j} + x^2y^5\boldsymbol{k}.$$

(a) Sketch the surface S, labelling clearly S_1, S_2, and ∂S.

(b) Evaluate

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S},$$

where S is orientated using the outward unit normal.

Problem 10. (12 marks) The divergence theorem in the plane can be expressed as

$$\int_C \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,ds \;=\; \iint_D \boldsymbol{\nabla}\cdot\boldsymbol{F}\,dx\,dy,$$

where D is a region in the x-y plane bounded by a simple closed curve C with positive orientation, $\boldsymbol{F}$ is a C^1 vector field, and $\hat{\boldsymbol{n}}$ is an outward pointing normal to C in the x-y plane.
Verify the divergence theorem in the plane for the vector field $\boldsymbol{F}$ given by $\boldsymbol{F}(x,y)=(4x,y)$, and the region $D=\left\{(x,y)\in\mathbb{R}\,|\,0\le y\le\sqrt{9-x^2}\right\}$.
Include a sketch of C and D clearly showing a unit normal vector $\hat{\boldsymbol{n}}$ and the orientation of C.

Problem 11. (12 marks) Using Gauss' Divergence theorem, calculate the flux integral

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S},$$

where $\boldsymbol{F}(x,y,z)=2x\boldsymbol{i}+y\boldsymbol{j}+z^2\boldsymbol{k}$, and S is the surface of the cone $z=2-\sqrt{x^2+y^2}$ that lies above the x-y plane. The surface is oriented so that the normal is pointing outwards.

Problem 12. (11 marks) Define *elliptic cylindrical* coordinates (u,v,w) by

$$x \;=\; a\cosh u\cos v, \qquad y \;=\; a\sinh u\sin v, \qquad z \;=\; w, \qquad (a>0)$$

where $u\ge 0$, $0\le v<2\pi$, and $-\infty<w<\infty$.

(a) Find the scale factors h_u, h_v, and h_w.

(b) Find the unit vectors $\boldsymbol{e}_u$, $\boldsymbol{e}_v$, and $\boldsymbol{e}_w$.

(c) If $f(u,v,w)=u^2$, determine $\nabla^2 f(u,v,w)$, expressing your answer in the simplest possible form.

4 Final Exam Semester 2 Year 2018

Problem 1. (10 marks) Consider the following function:

$$g(x,y) = \begin{cases} ye^{-1/x^2}, & \text{for } (x,y) \text{ with } x \neq 0, \\ y, & \text{for } (x,y) \text{ with } x = 0. \end{cases}$$

(a) Calculate $\lim\limits_{(x,y)\to(0,0)} g(x,y)$.

(b) Determine where g is continuous. Justify your answer, referring to any theorems you use.

(c) Using the definition of the partial derivative, calculate $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ at $(x,y) = (0,0)$.

Problem 2. (10 marks)

(a) Use a Lagrange multiplier to find the point (x,y,z) closest to the origin on the graph of the function $z = x + y - 3$. (Hint: to simplify work, take $f(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)$ as the function to be minimised.)

(b) Consider the general problem of finding points (x,y,z) which minimise the distance from the origin on the graph of a function $z = g(x,y)$. Give a system of equations in terms of $x, y, g, \dfrac{\partial g}{\partial x}$, and $\dfrac{\partial g}{\partial y}$ whose solutions will give such points.

Problem 3. (10 marks) A curve C has the parametric equations:

$$x = 2t, \quad y = t^2, \quad z = \log t, \qquad \text{for} \quad 0 < t < \infty.$$

(a) Find the acceleration $\mathbf{a}(t)$ and the unit tangent vector $\mathbf{T}(t)$ to C.

(b) Find the curvature of C at the point where $t = 1$.

(c) Let $\mathbf{N}(t)$ be the principal normal vector to C. The acceleration at the point $t = 1$ can be written as

$$\mathbf{a}(1) = a_T\mathbf{T}(1) + a_N\mathbf{N}(1). \qquad \text{(You do not need to prove this fact.)}$$

Show that $|\mathbf{a}(1)| = \sqrt{a_T^2 + a_N^2}$.

(d) Calculate a_T and a_N.

Problem 4. (10 marks) Let f be a scalar function of order C^2 on $\mathbb{R}^3$ and let $\mathbf{F}$ be a vector field of order C^2 on $\mathbb{R}^3$.

(a) Just using the definitions (i.e. without using the identities on the formula sheet), prove that $\mathrm{curl}(\mathrm{grad}(f)) = \mathbf{0}$.

(b) Just using the definitions (i.e. without using the identities on the formula sheet), prove that $\mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0$.

Problem 5. (10 marks) A triangle has vertices $(0,0,0)$, $(0,1,-1)$ and $(0,1,1)$. The plane of the triangle is rotated about the z-axis, and the moving triangle forms a solid (which is a cylinder from which two parts of a cone have been removed).

(a) Set up a multiple integral in *spherical coordinates* to calculate the volume of this solid.

(b) Find the volume.

Problem 6. (10 marks) Use multiple integration to find the moment of inertia about its axis of symmetry for a cylinder of radius a, height h, constant density μ and total mass M. Express your answer in terms of a and M.

Problem 7. (10 marks) Let C be the curve

$$\mathbf{c}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\,\mathbf{k}, \qquad \text{for} \quad 0 \le t \le 2\pi,$$

and let

$$\mathbf{F}(x,y,z) = 2x\,\mathbf{i} - 4yz^2\,\mathbf{j} - (4y^2z - 1)\,\mathbf{k}.$$

(a) Find f such that $\mathbf{F} = \boldsymbol{\nabla} f$.

(b) Calculate the work done by the vector field $\mathbf{F}$ to move a particle along the curve C in the direction of increasing t.

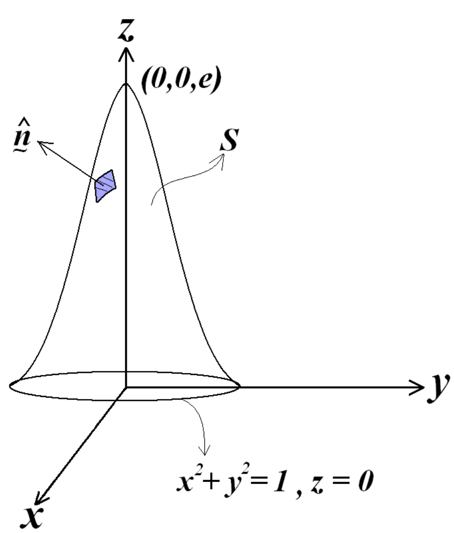
Problem 8. (10 marks) Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where S is the surface of the bell

$$z = (1-x^2-y^2)e^{1-x^2-y^2} \qquad \text{for } z \ge 0,$$

and

$$\mathbf{F}(x,y,z) = (e^y\cos z, (x^3+1)^{\frac{1}{2}}\sin z, x^2+y^2+3).$$

(Hint: Use the divergence theorem.)



Problem 9. (10 marks) Let the surface S be the disk $x^2 + y^2 \leq 9$ in the plane $z = 2$. The normal to S is directed upwards. Let $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$.

(a) Evaluate
$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S},$$
without using Stokes' theorem.

(b) Stokes' theorem asserts that the value of this surface integral is equal to the value of a certain line integral. Set up and evaluate the line integral.

Problem 10. (10 marks)

(a) Sketch the region enclosed by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $y = 0$ and $2y = x$, if $x > 0$.

(b) Use the change of variables
$$u = \frac{y}{x} \qquad \text{and} \qquad v = x^2 - y^2$$
to find the area of the region. (Hint: Note that $\dfrac{1}{1-u^2} = \dfrac{1}{2}\Big(\dfrac{1}{1+u} + \dfrac{1}{1-u}\Big)$.)

Problem 11. (10 marks) Use *spherical coordinates* for this question.

(a) Compute the scale factors h_r, h_θ, and h_φ for spherical coordinates.

(b) Show that the coordinate system is orthogonal.

(c) Find an expression for $\boldsymbol{\nabla}\theta$ in terms of r, θ, φ and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$.

(d) Find an expression for $\boldsymbol{\nabla} \cdot (\sin\theta\, \hat{\boldsymbol{\theta}})$ in terms of r, θ, φ and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$.

5 Final Exam Makeup Semester 2 Year 2018

Problem 1. (12 marks) Consider the following function:

$$f(x,y) = \begin{cases} \dfrac{2x^3+y^2}{x^2+3y^2}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0). \end{cases}$$

1. Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, if it exists.
2. Calculate $\dfrac{\partial f}{\partial x}$ if $(x,y)\neq(0,0)$.
3. Calculate $\dfrac{\partial f}{\partial x}$ if $(x,y)=(0,0)$.
4. Determine where f is continuous. Justify your answer.

Problem 2. (11 marks) Using Lagrange Multipliers, determine the critical points of the function

$$f(x,y,z) = xyz$$

subject to the constraint $2x+3y+z=6$.

Problem 3. (10 marks)

1. Let $\mathbf{T}$ and $\mathbf{B}$ be the unit tangent and unit binormal vectors to a path. Prove that
$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\cdot\mathbf{T} = 0.$$
2. Consider the vector field
$$\mathbf{F}(x,y) = (-x, 2y).$$
Derive the differential equation for the flow lines and solve the differential equation to obtain an expression for the flow lines in the form $y=f(x)$.

Problem 4. (15 marks)

1. Let $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^3$ be a C^1 vector field and $f:\mathbb{R}^3\to\mathbb{R}$ be a C^1 scalar function. Prove that
$$\boldsymbol{\nabla}\times(f\boldsymbol{F}) = f\boldsymbol{\nabla}\times\boldsymbol{F} + \boldsymbol{\nabla}f\times\boldsymbol{F}.$$
2. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r=|\mathbf{r}|$, $r\neq 0$. Calculate the following quantities, if they are defined:
$$\text{(i)}\boldsymbol{\nabla}\cdot\left(\frac{1}{r^2}\right); \qquad\qquad \text{(ii) } \nabla^2\left(\frac{1}{r^2}\right).$$

Problem 5. (11 marks) Let R be the solid capped cylinder that is bounded by the plane $z = 0$, the cylinder $x^2 + y^2 = 1$ and the hemisphere $z = \sqrt{9 - x^2 - y^2}$.

1. Sketch the region R.
2. Calculate the total mass of R if the mass per unit volume is $\mu = z^2$.

Problem 6. (9 marks) Let S be the surface of the torus represented by

$$x = (4 + \cos\phi)\cos\theta, \quad y = (4 + \cos\phi)\sin\theta, \quad z = \sin\phi$$

where $0 \le \phi \le 2\pi$ and $0 \le \theta \le 2\pi$.

1. Determine a normal vector to S.
2. Using part (a), determine the surface area of S.

Problem 7. (8 marks) Evaluate the line integral

$$\int_C (-2y^3 + \sinh^4 2x)\ dx + (2x^3 + \cosh^2 3y)\ dy$$

around the triangle in the x-y plane with vertices at $(0, 0)$, $(0, 2)$ and $(1, 2)$, traversed in the clockwise direction.

Problem 8. (18 marks)

1. State Gauss' theorem. Explain all symbols used and any required conditions.
2. Let S be the hemisphere $z = \sqrt{2 - x^2 - y^2}$. Using Gauss' theorem, evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

   where $\mathbf{F}(x, y, z) = 3y^3 z\,\mathbf{i} + (x - e^z)\,\mathbf{j} + \big(z^2 + 2\big)\,\mathbf{k}$ and S has an outward pointing normal.

Problem 9. (23 marks)

1. Let D be the semi-circular region

$$0 \le y \le \sqrt{9 - x^2}$$

   with a positively oriented boundary curve C. Sketch C and D, clearly showing an outward normal vector $\hat{\mathbf{n}}$ for C in the x-y plane and the orientation of C.
2. The divergence theorem in the plane can be written as

$$\int_{C=\partial D} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_D \boldsymbol{\nabla} \cdot \mathbf{F}\, dxdy.$$

   Verify the divergence theorem in the plane for the vector field $\mathbf{F}(x, y) = (4x, y^2)$ and the region D specified in part (a).

Problem 10. (8 marks) Define *parabolic cylindrical* coordinates (u, v, z) by

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv, \qquad z = z$$

where $u \in \mathbb{R},\ v \geq 0,\ z \in \mathbb{R}$ and $u^2 + v^2 > 0$.

1. Calculate the scale factors h_u, h_v, h_z.
2. Determine the unit vectors $\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_z$.
3. Evaluate

$$\boldsymbol{\nabla} \cdot (u^3\mathbf{e}_u + v^2 z\mathbf{e}_v).$$

6 Assignment 4 Semester 2 Year 2018

1. (a) Use

$$\sinh t = \frac{e^t - e^{-t}}{2}, \quad \cosh t = \frac{e^t + e^{-t}}{2}, \quad \tanh t = \frac{\sinh t}{\cosh t}, \quad \text{and} \quad \operatorname{sech} t = \frac{1}{\cosh t}$$

to show that $1 - \tanh^2 t = \operatorname{sech}^2 t$,

$$\frac{d\ \tanh t}{dt} = \operatorname{sech}^2\ t, \qquad \lim_{t\to 0} \tanh t = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \tanh t = 1.$$

(b) Use the change of variables $x = r\cos\theta$, $y = r\sin\theta$, $z = t - \tanh t$ to derive the formula

$$(\text{Volume of } V) = \iiint_V r \tanh^2 t dr\, d\theta\, dt$$

for any region V in $\mathbb{R}^3$.

(c) Let S be the parametrised surface (the pseudosphere) given by

$$\Phi(t,\theta) = \Big(\frac{\cos\theta}{\cosh t}, \frac{\sin\theta}{\cosh t}, t - \tanh t\Big), \qquad \text{for} \quad 0 \le t \text{ and } 0 \le \theta < 2\pi.$$

Let V be the region in $\mathbb{R}^3$ between the plane $z = 0$ and the surface S.

(ca) Graph the region V.

(cb) Compute the volume of the region V.

2. Let C be the directed curve forming the triangle $(0,0,0)$ to $(0,1,1)$ to $(1,1,1)$ to $(0,0,0)$.

(a) Let $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$. Is the field $\mathbf{F}$ conservative? Find $\int_C \mathbf{F}\cdot d\mathbf{s}$.

(b) Let $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$. Is the field $\mathbf{F}$ conservative? Find $\int_C \mathbf{F}\cdot d\mathbf{s}$.

3. Let $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and let S be the flat triangular surface with vertices $(0,0,0), (1,1,0), (2,2,2)$. Assume S is oriented towards the postive x-axis. Find

$$\iint_S \mathbf{F}\cdot d\mathbf{S}.$$

4. Let C be the curve $\mathbf{c}(t) = (t, \sin t)$, for $0 \le t \le \pi$. Use Green's theorem to evaluate the line integral

$$\int_C (3y\, dx + 2x\, dy).$$

7 Assignment 3 Semester 2 Year 2018

1. Let R be the region enclosed between $y = \frac{1}{2}x$, $y = 0$ and $x + y = 1$.
   Using the new coordinates $u = x + y$, $v = x - 2y$ evaluate the following double integral.
   $$\iint_R \sqrt{(x+y)(x-2y)}\, dx\, dy$$
   Sketch R both before and after the change of coordinates.

2. (a) Use
   $$\sinh t = \frac{e^t - e^{-t}}{2} \qquad \text{and} \qquad \cosh t = \frac{e^t + e^{-t}}{2}$$
   to show that $\cosh^2 t - \sinh^2 t = 1$,
   $$\frac{d\ \sinh t}{dt} = \cosh t, \quad \frac{d\ \cosh t}{dt} = \sinh t, \quad \lim_{t\to 0} \frac{1}{\cosh t} = 1, \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{\cosh t} = 0,$$
   (b) Let S be the parametrised surface (the pseudosphere) given by
   $$\Phi(t,\theta) = \left(\frac{\cos\theta}{\cosh t}, \frac{\sin\theta}{\cosh t}, t - \tanh t\right), \qquad \text{for} \quad 0 \le t \text{ and } 0 \le \theta < 2\pi.$$
   Calculate the surface area of S.

3. Let R be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$ and $z = x + y$.
   (a) Find the volume of R by integration.
   (b) Assume that the region is a solid of uniform density and use triple integration to find its center of mass.

8 Assignment 2 Semester 2 Year 2018

1. For $z = x^2 + y^3 - 6xy$, find the critical points and classify them by means of the second derivative test.

2. Consider the curve (a cardioid)
$$\mathbf{c}(t) = \big(8(1-\cos t)\cos t, 8(1-\cos t)\sin t, 1\big), \qquad \text{with } 0 < t < 2\pi.$$
(a) Graph the curve $\mathbf{c}$.

(b) Find the length of the arc $\mathbf{c}$.

3. Let n be an integer, let $r = (x^2 + y^2 + z^2)^{1/2}$ and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Assuming that $(x, y, z) \in \mathbb{R}^3 - \{(0,0,0)\}$, show that
$$\nabla(r^n) = nr^{n-2}\mathbf{r}.$$

(b) For which values of n is $\nabla^2(r^n) = 0$?

4. Consider the curve
$$\mathbf{c}(t) = \big(t, t^2, \tfrac{2}{3}t^3\big), \qquad \text{with } 0 \le t \le 2.$$

(a) Find the unit tangent vector $\mathbf{T}(t)$, the principal normal $\mathbf{N}(t)$ and the binormal $\mathbf{B}(t)$.

(b) Calculate the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the curve $\mathbf{c}$.

(c) Graph the curve $\mathbf{c}$.

9 Assignment 1 Semester 2 Year 2018

1. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by
$$f(x,y) = \begin{cases} \dfrac{3x^2 - y^2 - 2}{x^2 + y^2 - 2}, & \text{for } x^2 + y^2 \neq 2, \\ -1, & \text{for } x^2 + y^2 = 2. \end{cases}$$

(a) Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists or justify why the limit does not exist.

(b) Evaluate $\displaystyle\lim_{(x,y)\to(1,1)} f(x,y)$ if it exists or justify why the limit does not exist.

(c) Is f continuous at $(1,1)$. Justify your answer.

2. Consider the following function:
$$g(x,y) = \begin{cases} e^{-(x^2+y^2)} - 5, & \text{for } (x,y) \neq (0,0), \\ 1, & \text{for } (x,y) = (0,0). \end{cases}$$

(a) Graph the function $g(x,y)$.

(b) Calculate $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$.

(c) Determine where g is continuous. Justify your answer, referring to any theorems you use.

(d) Find $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ for all $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$.

(e) For which r is $g(x,y)$ a C^r function at $(0,1)$? (justify your answer)

3. Consider the functions $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $g\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$\begin{aligned} f(u,v) &= (2u^2, 3v, v^2 - u), \\ g(u,v,w) &= (v + w^2, u^2 + w), \\ h(u,v) &= (v^2 - u, 2u + v). \end{aligned}$$

Evaluate the derivative $\mathbf{D}(h(g(f(u,v))))$ at $(u,v) = (0,1)$ using the matrix version of the chain rule.

10 Assignment 4 Semester 2 Year 2017

1. Consider the parametrised surface defined by

$$x = u\sin^3 v, \quad y = u\cos^3 v, \quad z = u, \qquad \text{where } u \geq 0 \text{ and } 0 \leq v \leq 2\pi.$$

   (a) Find a vector normal to the surface in terms of u and v.
   (b) For what values of u and v is the surface smooth?
   (c) Find the equation of the tangent plane to the surface at $(\sqrt{2}, -\sqrt{2}, 4)$.
   (d) Find the Cartesian equation of this surface.

2. Find the area of the surface of that portion of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 2x$.

3. Calculate the flux of each of the vector fields

$$\mathbf{F}(x, y, z) = z^4\mathbf{k} \qquad \text{and} \qquad \mathbf{G}(x, y, z) = z^5\mathbf{k}$$

   across the unit sphere oriented with the outward pointing normal.

11 Assignment 3 Semester 2 Year 2017

1. Consider the double integral below:

$$\int_0^{27}\int_{\sqrt[3]{y}}^{3} \sinh(x^2)dxdy.$$

   (a) Sketch the region of integration, labelling all its vertices.
   (b) Calculate the double integral by interchanging the order of integration.

2. Consider the solid V of unit density, where V is the region inside the sphere $x^2+y^2+z^2 = 4\cdot 6352^2$ and outside the cylinder $x^2 + y^2 = 6352^2$.

   (a) Using spherical coordinates, set up and calculate the moment of inertia for V about the z-axis. Show all calculations.
   (b) Now, using cylindrical coordinates, set up and calculate the moment of inertia of V about the z-axis. Show all calculations.

3. Let

$$\mathbf{F}(x, y, z) = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k}).$$

   Find the work done by the force $\mathbf{F}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along each of the following curves:

(a) along the straight line $x = y = z$;

(b) along the curve $x = t$, $y = t^2$, $z = t^4$;

(c) along the x-axis to $(1, 0, 0)$, then in a straight line to $(1, 1, 0)$ and from there in a straight line to $(1, 1, 1)$.

Is the force $\mathbf{F}$ conservative?

12 Assignment 2 Semester 2 Year 2017

1. The cone $z^2 = x^2 + y^2$ is cut by the plane $z = 1 + x + y$ in a curve C. Find the points on C that are nearest to, and furthest from, the origin.

2. Consider the curve

$$\mathbf{c}(t) = \big(2(-t + \sin t), \sqrt{3}(1 - \cos t), 1 - \cos t\big), \qquad \text{with } 0 < t < 2\pi.$$

(a) Find the length of the arc $\mathbf{c}$.

(b) Prove that the unit tangent vector is given by

$$\mathbf{T}(t) = \left(-\sin(t/2), \frac{\sqrt{3}\cos(t/2)}{2}, \frac{\cos(t/2)}{2}\right),$$

and therefore find the principal normal $\mathbf{N}(t)$ and the binormal $\mathbf{B}(t)$, for $0 < t < 2\pi$.

(c) Calculate the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the curve $\mathbf{c}$. What geometric conclusion can you draw from your result?

3. Let

$$\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}.$$

Is $\mathbf{F}$ irrotational? If so, find a scalar function f such that $\mathbf{F} = \nabla f$.

13 Assignment 1 Semester 2 Year 2017

1. Consider the function $f\colon \{(x, y) \in \mathbb{R}^2 \mid x \neq y\} \to \mathbb{R}$ given by

$$f(x, y) = \frac{x^2 - y^2}{x^3 - y^3}.$$

(a) Evaluate $\lim_{(x,y)\to(0,0)} f(x, y)$ if it exists or justify why the limit does not exist.

(b) Evaluate $\lim_{(x,y)\to(1,1)} f(x, y)$ if it exists or justify why the limit does not exist.

(c) Is f continuous at $(1, 1)$. Justify your answer.

2. Consider the following function:

$$g(x,y) = \begin{cases} \dfrac{x^2 y}{3x^2 + 4y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) Calculate $\lim_{(x,y)\to(0,0)} g(x,y)$.

(b) Determine where g is continuous. Justify your answer, referring to any theorems you use.

(c) Find $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ for all $(x,y) \in \mathbb{R}^2$.

(d) Determine where g is C^1. Justify your answer, referring to any theorems you use.

3. Consider the function

$$h(x,y) = \sqrt{35 - 6x^2 - 2y^2}.$$

(a) What is an appropriate point (a,b), where a and b are integers, about which to approximate $h(2.02, -0.96)$ using a Taylor series?

(b) Using the (a,b) you found in Q3a, estimate $h(2.02, -0.96)$ with a linear Taylor polynomial.