School of Mathematics and Statistics
MAST20009 Vector Calculus, Semester 2 2018
Assignment 2 and Cover Sheet

Student Name	*Student Number*
Tutor's Name	*Tutorial Day/Time*

Submit your assignment to your tutor's MAST20009 assignment box before 11am on Tuesday 4th September.

This assignment is worth 5% of your final MAST20009 mark. You must attach this cover sheet to your assignment.

Note:
- Full working must be shown in your solutions.
- Assignments must be neatly handwritten in blue or black pen. Diagrams can be drawn in pencil.
- Marks will be deducted for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- There are four questions in this assignment.

1. For $z = x^2 + y^3 - 6xy$, find the critical points and classify them by means of the second derivative test.

2. Consider the curve (a cardioid)

$$\mathbf{c}(t) = \big(8(1-\cos t)\cos t, 8(1-\cos t)\sin t, 1\big), \qquad \text{with } 0 < t < 2\pi.$$

(a) Graph the curve $\mathbf{c}$.

(b) Find the length of the arc $\mathbf{c}$.

3. Let n be an integer, let $r = (x^2+y^2+z^2)^{1/2}$ and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Assuming that $(x, y, z) \in \mathbb{R}^3 - \{(0,0,0)\}$, show that

$$\nabla(r^n) = nr^{n-2}\mathbf{r}.$$

(b) For which values of n is $\nabla^2(r^n) = 0$?

4. Consider the curve

$$\mathbf{c}(t) = \big(t, t^2, \tfrac{2}{3}t^3\big), \qquad \text{with } 0 \le t \le 2.$$

(a) Find the unit tangent vector $\mathbf{T}(t)$, the principal normal $\mathbf{N}(t)$ and the binormal $\mathbf{B}(t)$.

(b) Calculate the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the curve $\mathbf{c}$.

(c) Graph the curve $\mathbf{c}$.

(1) For $z = x^2 + y^3 - 6xy$ find the critical points and classify them by the second derivative test.

<u>Solution:</u> $f(x,y) = x^2 + y^3 - 6xy$.

The critical points are when

$$0 = \vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x - 6y, 3y^2 - 6x)$$

So $\begin{matrix} 2x - 6y = 0 \\ 3y^2 - 6x = 0 \end{matrix}$ So $\begin{matrix} x = 3y \\ 3y^2 - 6 \cdot 3y = 0. \end{matrix}$ So $\begin{matrix} x = 3y \\ y(y-6) = 0. \end{matrix}$

So $y = 0$ or $y = 6$.

So $(x,y) = (0,0)$ and $(x,y) = (18,6)$ are critical points.

Then $\frac{\partial^2 f}{\partial x^2} = 2$ and

$$\det(Hf) = \det\begin{pmatrix} 2 & -6 \\ -6 & 6y \end{pmatrix} = 12y - 36.$$

So $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(0,0)} = 2 \in \mathbb{R}_{>0}$ and $\left.\det Hf\right|_{(x,y)=(0,0)} = 12 \cdot 0 - 36 = -36 \in \mathbb{R}_{<0}$

So $(0,0)$ is a saddle point.

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Since

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)=(18,6)} = 2 \text{ and } \det Hf\Big|_{(x,y)=(18,6)} = 12\cdot 6 - 36 = 12\cdot 3 = 36 \in \mathbb{R}_{>0}$$

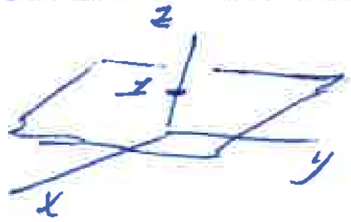
then $(18,6)$ is a minimum

(2) The curve is

$$\vec{c}(t) = (8(1-\cos t)\cos t,\ 8(1-\cos t)\sin t,\ 1)$$

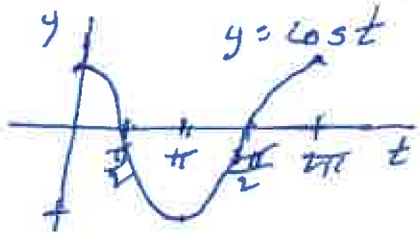
(a) Graph the curve.

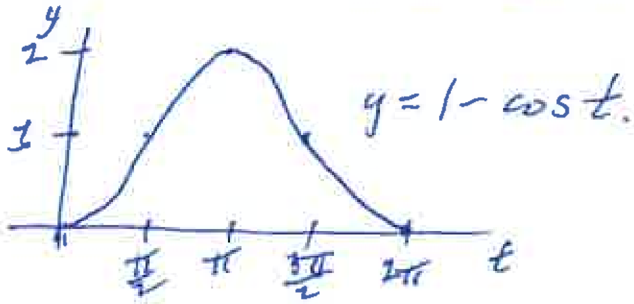
Since $z=1$ this curve is completely on the $z=1$ plane



If $x = 8(1-\cos t)\cos t$
$y = 8(1-\cos t)\sin t$ then $x^2+y^2 = 8^2(1-\cos t)^2$

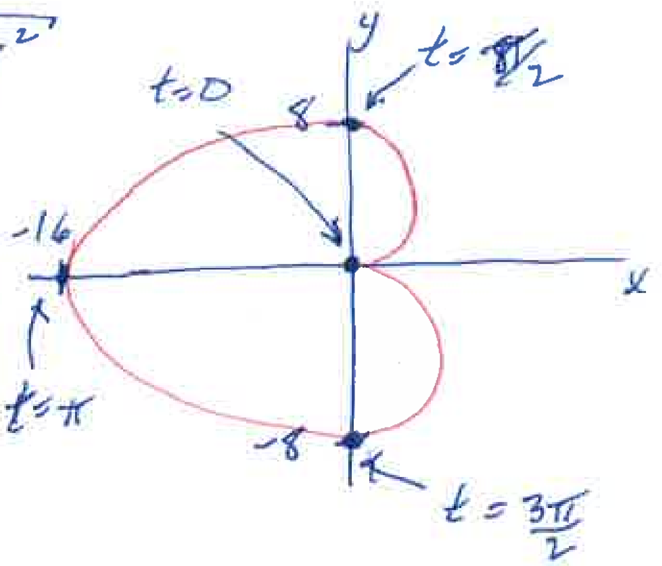
so that the point at time t is distance $8|1-\cos t|$ from the z-axis.



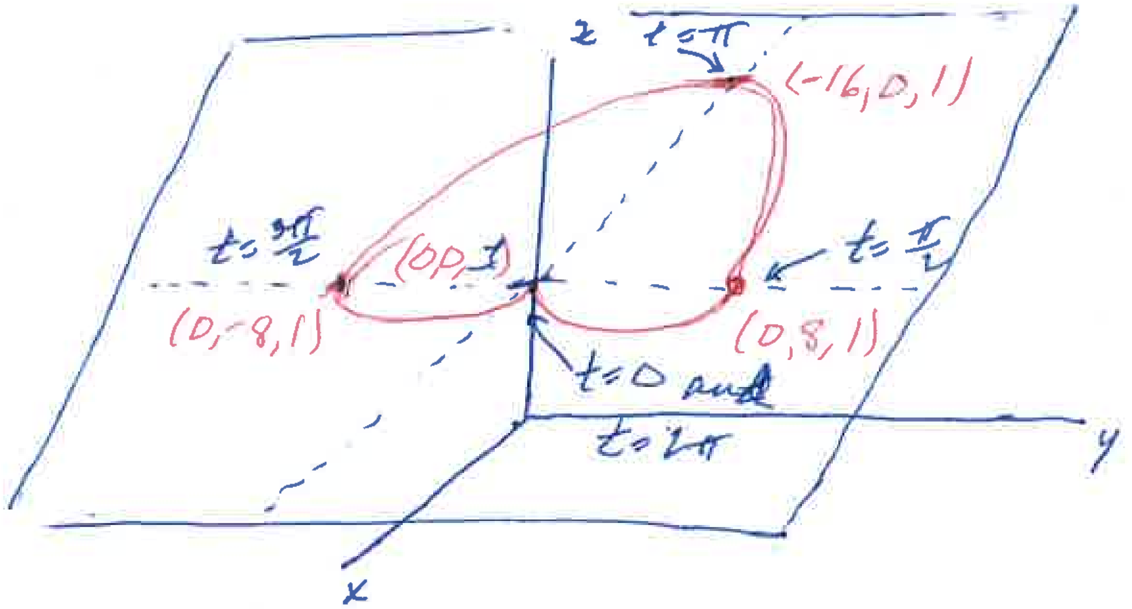


Some points on the curve $\vec{c}$:

	x	y	$\sqrt{x^2+y^2}$
$t=0$	$8\cdot0\cdot1$	$8\cdot0\cdot0$	0
$t=\frac{\pi}{2}$	$8\cdot1\cdot0$	$8\cdot1\cdot1$	8
$t=\pi$	$8\cdot2\cdot(-1)$	$8\cdot2\cdot0$	16
$t=\frac{3\pi}{2}$	$8\cdot1\cdot0$	$8\cdot1(-1)$	8
$t=2\pi$	$8\cdot0\cdot1$	$8\cdot0\cdot0$	



So the graph of $\vec{c}(t)$ is:



(b) $\vec{c} = (8(1-\cos t)\cos t,\ 8(1-\cos t)\sin t,\ 1)$

$= 8(\cos t - \cos^2 t,\ \sin t - \sin t\cos t,\ \frac{1}{8})$

So

$$\frac{d\vec{c}}{dt} = 8(-\sin t - 2\cos t(-\sin t),\ \cos t - \cos t\cos t - \sin t(-\sin t),\ 0)$$

$$= 8(2\sin t\cos t - \sin t,\ \cos t - (\cos^2 t - \sin^2 t),\ 0)$$

$$= 8(\sin 2t - \sin t,\ \cos t - \cos 2t,\ 0)$$

So

$$\left|\frac{d\vec{c}}{dt}\right| = 8\sqrt{(\sin 2t - \sin t)^2 + (\cos t - \cos 2t)^2 + 0^2}$$

$$= 8\sqrt{\sin^2 2t - 2\sin t\sin 2t + \sin^2 t + \cos^2 t - 2\cos t\cos 2t + \cos^2 2t}$$

$$= 8\sqrt{1 - 2\sin t \sin 2t + 1 - 2\cos t \cos 2t}$$

$$= 8\sqrt{2 - 2(\cos t \cos 2t + \sin t \sin 2t)}$$

$$= 8\sqrt{2}\sqrt{1 - (\cos(-t)\cos 2t - \sin(-t)\sin 2t}$$

$$= 8\sqrt{2}\sqrt{1 - \cos(-t + 2t)}$$

$$= 8\sqrt{2}(1 - \cos t)^{\frac{1}{2}}$$

So arclength from $t=0$ to $t=2\pi$ $= \displaystyle\int_{t=0}^{t=2\pi} 8\sqrt{2}(1-\cos t)^{\frac{1}{2}}\,dt$

$$= 8\sqrt{2}\int_{t=0}^{t=2\pi} \frac{(1-\cos t)^{\frac{1}{2}}(1+\cos t)^{\frac{1}{2}}}{(1+\cos t)^{\frac{1}{2}}}\,dt$$

$$= 8\sqrt{2}\int_{t=0}^{2\pi} \frac{(1-\cos^2 t)^{\frac{1}{2}}}{(1+\cos t)^{\frac{1}{2}}}\,dt$$

$$= 8\sqrt{2}\int_{t=0}^{t=2\pi} \frac{(\sin^2 t)^{\frac{1}{2}}}{(1+\cos t)^{\frac{1}{2}}}\,dt = 8\sqrt{2}\int_{t=0}^{t=2\pi} \frac{|\sin t|}{(1+\cos t)^{\frac{1}{2}}}\,dt$$

$$= 8\sqrt{2}\int_{t=0}^{\pi} \frac{\sin t}{(1+\cos t)^{\frac{1}{2}}}\,dt + 8\int_{t=\pi}^{t=2\pi} \frac{\sqrt{2}(-\sin t)}{(1+\cos t)^{\frac{1}{2}}}\,dt$$

$$= 8\sqrt{2}\left(-2(1+\cos t)^{\frac{1}{2}}\right)\Big]_{t=0}^{t=\pi}$$

$$+ 8\sqrt{2}\left(2(1+\cos t)^{\frac{1}{2}}\right)\Big]_{t=\pi}^{t=2\pi}$$

$$= 8\sqrt{2}\left((-2(1-1)^{\frac{1}{2}}) - (-2(1+1)^{\frac{1}{2}})\right)$$

$$+ 8\sqrt{2}\left(2(1+1)^{\frac{1}{2}} - 2(1-1)^{\frac{1}{2}}\right)$$

$$= 8\sqrt{2}\left(0+2\sqrt{2}\right) + 8\sqrt{2}\left(2\sqrt{2}-0\right)$$

$$= 2\cdot 8\sqrt{2}\,2\sqrt{2} = 2\cdot 8\cdot 2\cdot 2 = 8\cdot 8 = 64.$$

Alternatively,

$$\left|\frac{d\vec{c}}{dt}\right| = 8\sqrt{2}\,(1-\cos t)^{\frac{1}{2}}$$

$$= 8\sqrt{2}\left(1-\cos\left(\tfrac{t}{2}+\tfrac{t}{2}\right)\right)^{\frac{1}{2}}$$

$$= 8\sqrt{2}\left(1-\left(\cos^2\left(\tfrac{t}{2}\right)-\sin^2\left(\tfrac{t}{2}\right)\right)\right)^{\frac{1}{2}}$$

$$= 8\sqrt{2}\left(1-\cos^2\left(\tfrac{t}{2}\right)+\sin^2\left(\tfrac{t}{2}\right)\right)^{\frac{1}{2}}$$

$$= 8\sqrt{2}\left(\sin^2\left(\tfrac{t}{2}\right)+\sin^2\left(\tfrac{t}{2}\right)\right)^{\frac{1}{2}}$$

$$= 8\sqrt{2}\left(2\sin^2\left(\tfrac{t}{2}\right)\right)^{\frac{1}{2}} = 8\sqrt{2}\sqrt{2}\left|\sin\left(\tfrac{t}{2}\right)\right|$$

$$= 16\left|\sin\left(\tfrac{t}{2}\right)\right|.$$

So arclength from $t=0$ to $t=2\pi$ $= \int_0^{2\pi} 16\left|\sin\left(\tfrac{t}{2}\right)\right| dt$

$$= \int_0^{2\pi} 16\sin\left(\tfrac{t}{2}\right) dt, \quad \text{since } \sin\left(\tfrac{t}{2}\right) \geq 0 \text{ for } 0 \leq t \leq 2\pi$$

$$= -16\cos\left(\tfrac{t}{2}\right)\cdot 2\Big]_{t=0}^{t=2\pi}$$

$$= -(32\cos\left(\tfrac{2\pi}{2}\right) - (-32\cos\left(\tfrac{0}{2}\right))$$

$$= -32\cdot(-1) + 32\cdot 1 = 32+32 = 64$$

Ass 2 Q3 ①

Vector calculus 2018 Assignment 2

(3) Let $n \in \mathbb{Z}$. Let

$$r = (x^2+y^2+z^2)^{\frac{1}{2}} \quad \text{and} \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

(a) Show that $\vec{\nabla}(r^n) = n r^{n-2} \vec{r}$.

Solution: $r^n = (x^2+y^2+z^2)^{n/2}$

$$\vec{\nabla}(r^n) = \frac{\partial r^n}{\partial x}\hat{i} + \frac{\partial r^n}{\partial y}\hat{j} + \frac{\partial r^n}{\partial z}\hat{k}$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot 2x\,\hat{i}$$

$$+ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} 2y\,\hat{j}$$

$$+ \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} 2z\,\hat{k}$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}}(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$= n r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = n r^{n-2}\vec{r}$$

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(3b) For what n is $\nabla^2(r^n)=0$?

Solution: First note that if $\vec{F}=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ then

$$\vec{\nabla}\cdot(f\vec{F})=f(\vec{\nabla}\cdot\vec{F})+(\vec{\nabla}f)\cdot\vec{F}$$

since

$$\vec{\nabla}\cdot(f\vec{F})=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot(fF_1\hat{i}+fF_2\hat{j}+fF_3\hat{k})$$

$$=\frac{\partial fF_1}{\partial x}+\frac{\partial fF_2}{\partial y}+\frac{\partial fF_3}{\partial z}$$

$$=f\frac{\partial F_1}{\partial x}+\frac{\partial f}{\partial x}F_1+f\frac{\partial F_2}{\partial y}+\frac{\partial f}{\partial y}F_2+f\frac{\partial F_3}{\partial z}+\frac{\partial f}{\partial z}F_3$$

$$=f\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)+\left(\frac{\partial f}{\partial x}F_1+\frac{\partial f}{\partial y}F_2+\frac{\partial f}{\partial z}F_3\right)$$

$$=f(\vec{\nabla}\cdot\vec{F})+\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)\cdot(F_1\hat{i}+F_2\hat{j}+F_3\hat{k})$$

$$=f(\vec{\nabla}\cdot\vec{F})+(\vec{\nabla}f)\cdot\vec{F}.$$

Then

$$\nabla^2(r^n)=\vec{\nabla}\cdot(\vec{\nabla}r^n)$$

$$=\vec{\nabla}\cdot(nr^{n-2}\vec{r}),\ \text{by part (a)}$$

$$=nr^{n-2}(\vec{\nabla}\cdot\vec{r})+(\vec{\nabla}(nr^{n-2}))\cdot\vec{r}$$

$$=nr^{n-2}\left(\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k})\right)$$

$$+n(n-2)r^{n-4}\vec{r}\cdot\vec{r}$$

$$= n r^{n-2}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)$$

$$+ n(n-2) r^{n-4}\left((x\hat{i}+y\hat{j}+z\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})\right)$$

$$= n r^{n-2}\cdot 3 + n(n-2)\, r^{n-4}(x^2+y^2+z^2)$$

$$= 3n\, r^{n-2} + n(n-2)\, r^{n-2}$$

$$= 3n\, r^{n-2} + n(n-2)\, r^{n-2} = r^{n-2}(3n + n^2 - 2n)$$

$$= r^{n-2}(n^2+n) = r^{n-2}\, n(n+1)$$

$$= n(n+1)\,(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

So $\nabla^2(r^n) = n(n+1)\, r^{n-2}$

When $(x,y,z) \neq (0,0,0)$ then $(x^2+y^2+z^2) \neq 0$ and so

$\nabla^2(r^n) = 0$ only when $n=0$ or $n=-1$.

(4) Let

$$\vec{c}(t) = (t, t^2, \tfrac{2}{3}t^3) \quad \text{for} \quad 0 \le t \le 2$$

Compute $\vec{T}(t)$, $\vec{N}(t)$, $\vec{B}(t)$, $\kappa(t)$, and $\tau(t)$.

Solution

$$\frac{d\vec{c}}{dt} = (1, 2t, 2t^2) \text{ and}$$

$$\left|\frac{d\vec{c}}{dt}\right| = (1^2 + (2t)^2 + (2t^2)^2)^{\frac{1}{2}} = (1 + 4t^2 + 4t^4)^{\frac{1}{2}}$$

$$= ((1+2t^2)^2)^{\frac{1}{2}} = 1 + 2t^2.$$

So $\vec{T}(t) = \dfrac{1}{1+2t^2}(1, 2t, 2t^2)$

Then

$$\frac{d\vec{T}}{dt} = \frac{1}{1+2t^2}(0, 2, 4t) + \frac{(-1)\cdot 4t}{(1+2t^2)^2}(1, 2t, 2t^2)$$

$$= \frac{(1+2t^2)(0, 2, 4t) - 4t(1, 2t, 2t^2)}{(1+2t^2)^2}$$

$$= \frac{(0, 2+4t^2, 4t+8t^3) + (-4t, -8t^2, -8t^3)}{(1+2t^2)^2}$$

$$= \frac{1}{(1+2t^2)^2}(-4t, 2-4t^2, 4t)$$

So
$$\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{(1+2t^2)^2}\sqrt{16t^2+(2-4t^2)^2+16t^2}$$

$$= \frac{1}{(1+2t^2)^2}\sqrt{32t^2+4-16t^2+16t^4}$$

$$= \frac{1}{(1+2t^2)^2}\sqrt{16t^4+16t^2+4}$$

$$= \frac{2}{(1+2t^2)^2}\sqrt{4t^4+4t^2+1}$$

$$= \frac{2}{(1+2t^2)^2}\sqrt{(2t^2+1)^2} = \frac{2(1+2t^2)}{(1+2t^2)^2} = \frac{2}{1+2t^2}$$

So
$$\vec{N}(t) = \frac{\frac{1}{(1+2t^2)^2}(-4t, 2-4t^2, 4t)}{\frac{2}{1+2t^2}}$$

$$= \frac{1}{1+2t^2}(-2t, 1-2t^2, 2t)$$

So
$$\kappa(t) = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{d\vec{c}}{dt}\right|} = \frac{\frac{2}{1+2t^2}}{1+2t^2} = \frac{2}{(1+2t^2)^2}$$

Then

$$\vec{B}(t) = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 2t^2 \\ -2t & 1-2t^2 & 2t \end{vmatrix} \frac{1}{(1+2t^2)(1+2t^2)}$$

$$= \begin{pmatrix} \hat{i}\,(4t^2 - 2t^2 + 4t^4) \\ -\hat{j}\,(2t + 4t^3) \\ +\hat{k}\,(1 - 2t^2 + 4t^2) \end{pmatrix} \frac{1}{(1+2t^2)^2}$$

$$= \frac{1}{(1+2t^2)^2}\left(4t^4 + 2t^2,\ -4t^3 - 2t,\ 2t^2 + 1\right)$$

$$= \frac{(2t^2+1)\,(2t^2, -2t, 1)}{(1+2t^2)^2} = \frac{1}{1+2t^2}(2t^2, -2t, 1)$$

The 3rd coordinate of $\frac{d\vec{B}}{dt}$ is $\frac{d}{dt}(1+2t^2)^{-1} = -4t(1+2t^2)^{-2}$

The 3rd coordinate of $\dfrac{\frac{d\vec{B}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$ is $\dfrac{-4t(1+2t^2)^{-2}}{1+2t^2} = \dfrac{-4t}{(1+2t^2)^3}$

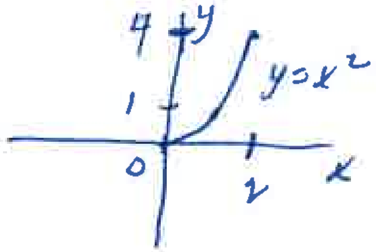
The 3rd coordinate of $\vec{N}$ is $\dfrac{2t}{1+2t^2}$

So $\dfrac{-4t}{(1+2t^2)^3} = -\tau(t)\,\dfrac{2t}{1+2t^2}$. So $\tau(t) = \dfrac{2}{(1+2t^2)^2}$

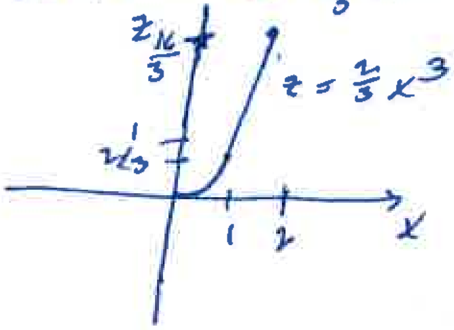
Vector Calculus 2018 Assignment 2 Ass 2 Q4 (4)

Graph the curve

$(x,y) = (t, t^2)$ is $y = x^2$



$(x,z) = (t, \frac{2}{3}t^3)$ is $z = \frac{2}{3}x^3$



So $(x,y,z) = (t, t^2, t^3)$ looks like