MAST20009 Assignment 4 Solutions

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(1) The surface S has

$$x(u,v) = u\sin^3 v, \quad y(u,v) = u\cos^3 v, \quad z(u,v) = u.$$

(a) So $\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\sin^3 v, \cos^3 v, 1)$

$$\vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (3u\sin^2 v\cos v, -3u\cos^2 v\sin v, 0)$$

Then

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin^3 v & \cos^3 v & 1 \\ 3u\sin^2 v\cos v & -3u\cos^2 v\sin v & 0 \end{vmatrix}$$

$$= \hat{\imath}(0 + 3u\cos^2 v\sin v) - \hat{\jmath}(0 - 3u\sin^2 v\cos v) + \hat{k}(-3u\sin^4 v\cos^2 v - 3u\sin^2 v\cos^4 v)$$

$$= (3u\cos^2 v\sin v, 3u\sin^2 v\cos v, -3u\sin^2 v\cos^2 v)$$

is a vector normal to the surface S.

(b) The surface is smooth when $\vec{T}_u \times \vec{T}_v \neq 0$.

This is when

$$u \neq 0 \text{ and } \sin v \neq 0 \text{ and } \cos v \neq 0.$$

This is when

$$u \neq 0 \text{ and } v \neq 0, v \neq \pi, v \neq 2\pi, v \neq \frac{\pi}{2}, v \neq \frac{3\pi}{2}.$$

So S is smooth when

$$u \neq 0 \text{ and } v \notin \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}.$$

(c) $(x_0, y_0, z_0) = (\sqrt{2}, -\sqrt{2}, 4) = \left(4\left(\frac{1}{\sqrt{2}}\right)^3, 4\left(\frac{-1}{\sqrt{2}}\right)^3, 4\right)$

$= \left(4\sin^3\left(\frac{3\pi}{4}\right), 4\cos^3\left(\frac{3\pi}{4}\right), 4\right)$

So $(x_0, y_0, z_0) = (\sqrt{2}, -\sqrt{2}, 4)$ when $u_0 = 4$, $v_0 = \frac{3\pi}{4}$.

The equation of the tangent plane is

$0 = (x - x_0, y - y_0, z - z_0) \cdot \vec{n}(u_0, v_0)$

$= (x - \sqrt{2}, y + \sqrt{2}, z - 4) \cdot (\vec{T}_u \times \vec{T}_v)(u_0, v_0)$

$= (x - \sqrt{2}, y + \sqrt{2}, z - 4) \cdot (\vec{T}_u \times \vec{T}_v)\left(4, \frac{3\pi}{4}\right)$

$= (x - \sqrt{2}, y + \sqrt{2}, z - 4) \cdot \left(3\cdot 4\left(\frac{-1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right), 3\cdot 4\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{-1}{\sqrt{2}}\right), -3\cdot 4\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{-1}{\sqrt{2}}\right)^2\right)$

$= (x - \sqrt{2}, y + \sqrt{2}, z - 4) \cdot (3\sqrt{2}, -3\sqrt{2}, -3)$

$= 3\sqrt{2}(x - \sqrt{2}) - 3\sqrt{2}(y + \sqrt{2}) - 3(z - 4)$

$= 3\sqrt{2}x - 6 - 3\sqrt{2}y - 6 - 3z + 12$

$= 3\sqrt{2}x - 3\sqrt{2}y - 3z.$

So the equation of the tangent plane is

$\sqrt{2}x - \sqrt{2}y - z = 0.$

(d) $x = u\sin^3 v$

$y = u\cos^3 v$ and so

$z = u$

$x^{3/2} = u^{3/2}\sin^2 v$

$y^{3/2} = u^{3/2}\cos^2 v$

$z^{3/2} = u^{3/2}$.

So $x^{3/2} + y^{3/2} = z^{3/2}$ is the Cartesian equation of the surface S.

(2) The sphere $x^2+y^2+z^2=4$ is parametrized

by $\Phi(u,v) = (u, v, (4-u^2-v^2)^{\frac{1}{2}})$

Then
$$\vec{T}_u = \left(1, 0, -2u\tfrac{1}{2}(4-u^2-v^2)^{-\frac{1}{2}}\right) = \left(1, 0, \frac{-u}{\sqrt{4-u^2-v^2}}\right)$$

$$\vec{T}_v = \left(0, 1, -2v\tfrac{1}{2}(4-u^2-v^2)^{-\frac{1}{2}}\right) = \left(0, 1, \frac{-v}{\sqrt{4-u^2-v^2}}\right)$$

Then
$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{-u}{\sqrt{4-u^2-v^2}} \\ 0 & 1 & \frac{-v}{\sqrt{4-u^2-v^2}} \end{vmatrix}$$

$$= \hat{i}\left(0+\frac{u}{\sqrt{4-u^2-v^2}}\right) - \hat{j}\left(\frac{-v}{\sqrt{4-u^2-v^2}} - 0\right) + \hat{k}(1-0)$$

$$= \left(\frac{u}{\sqrt{4-u^2-v^2}}, \frac{v}{\sqrt{4-u^2-v^2}}, 1\right)$$

and

$$|\vec{T}_u \times \vec{T}_v| = \sqrt{\frac{u^2}{4-u^2-v^2} + \frac{v^2}{4-u^2-v^2} + 1^2}$$

$$= \sqrt{\frac{u^2+v^2+4-u^2-v^2}{4-u^2-v^2}} = \frac{2}{\sqrt{4-u^2-v^2}}$$

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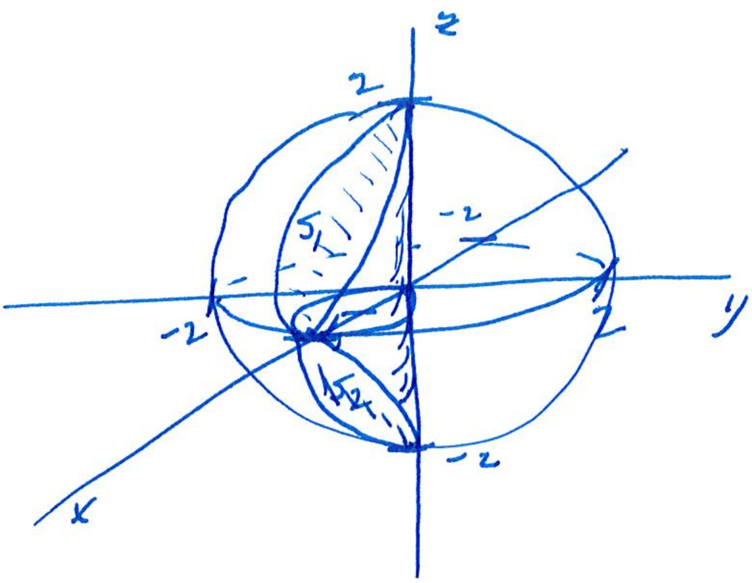


$S = S_1 \cup S_2$

$$A(S) = \iint_S dS = 2\iint_{S_1} dS = 2\iint_D |\vec{T}_u \times \vec{T}_v|\, du\, dv$$

$$= 2\iint_D \frac{2}{\sqrt{4-u^2-v^2}}\, du\, dv = 2\iint_D \frac{2}{(4-r^2)^{\frac{1}{2}}}\, r\, dr\, d\theta$$

$$= 2\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2\cos\theta} 2r(4-r^2)^{-\frac{1}{2}}\, dr\, d\theta$$

since the curve $x^2+y^2=2x$ is

$r^2 = 2r\cos\theta$ when $x = r\cos\theta$
$y = r\sin\theta$

and $r^2 - 2r\cos\theta = 0$ is $r(r-2\cos\theta) = 0$

So $A(S) = 2\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2\cos\theta} 2r(4-r^2)^{-\frac{1}{2}}\, dr\, d\theta$

$= 2\int_{\theta=\frac{\pi}{2}}^{\theta=\pi/2} -2(4-r^2)^{\frac{1}{2}}\Big|_{r=0}^{r=2\cos\theta} d\theta$

$= -4\int_{\theta=-\frac{\pi}{2}}^{\theta=\pi/2} \left((4-4\cos^2\theta)^{\frac{1}{2}} - 4^{\frac{1}{2}}\right) d\theta$

$= -4\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} (2|\sin\theta| - 2)\, d\theta = -8\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} (|\sin\theta| - 1)\, d\theta$

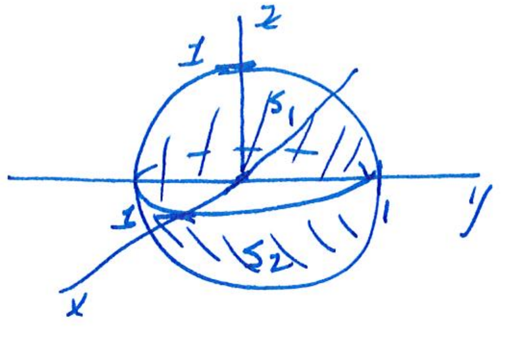
$= 2(-8)\int_{\theta=0}^{\theta=\frac{\pi}{2}} (\sin\theta - 1)\, d\theta = -16(-\cos\theta - \theta)\Big|_{\theta=0}^{\theta=\pi/2}$

$= -16\left(-\cos\frac{\pi}{2} - \frac{\pi}{2} - (-\cos 0 - 0)\right)$

$= -16\left(0 - \frac{\pi}{2} + 1\right) = 8\pi - 16.$

So the area of the surface S is $8\pi - 16$.

(3)



$S = S_1 \cup S_2$

Flux across $S = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \hat{n}\, dS$

$= \iint_S \vec{F}\cdot(\vec{T}_u \times \vec{T}_v)\, du\, dv.$

The sphere S is parametrized by

$S_1 = \Phi_1(u,v) = (u, v, \sqrt{1-u^2-v^2})$

$S_2 = \Phi_2(u,v) = (u, v, -\sqrt{1-u^2-v^2})$

For the surface S_1:

$\vec{T}_u = \left(1, 0, \frac{1}{2}(1-u^2-v^2)^{-\frac{1}{2}}(-2u)\right) = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right)$

$\vec{T}_v = \left(0, 1, \frac{1}{2}(1-u^2-v^2)^{-\frac{1}{2}}(-2v)\right) = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right)$

and

$\vec{T}_u \times \vec{T}_v = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right)$

For the surface S_2 a similar computation gives

$\vec{T}_u \times \vec{T}_v = \left(\frac{-u}{\sqrt{1-u^2-v^2}}, \frac{-v}{\sqrt{1-u^2-v^2}}, 1\right)$

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For S_1 the outward pointing normal is ②

$$\vec{n} = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right)$$

For S_2 the outward pointing normal is

$$\vec{n} = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, -1\right)$$

If

$u = r\cos\theta$
$v = r\sin\theta$ then $u^2+v^2=r^2$ and $du\,dv = r\,dr\,d\theta$

$z = (1-r^2)^{\frac{1}{2}}$ for the surface S_1

$z = -(1-r^2)^{-\frac{1}{2}}$ for the surface S_2

(a) Let $\vec{F} = z^4\hat{k}$. Then

$$\text{Flux across } S = \iint_S \vec{F}\cdot\vec{n}\,du\,dv$$

$$= \iint_{S_1} \vec{F}\cdot\vec{n}\,du\,dv + \iint_{S_2} \vec{F}\cdot\vec{n}\,du\,dv$$

$$= \iint_{S_1} z^4\,du\,dv + \iint_{S_2} -z^4\,du\,dv$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (1-r^2)^2\,r\,dr\,d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} -(1-r^2)^2\,r\,dr\,d\theta$$

$$= 0.$$

(b) Let $\vec{F} = z^5\hat{k}$. Then

$$\text{Flux across } S = \iint_S F \cdot \hat{n}\, du\, dv$$

$$= \iint_{S_1} \vec{F}\cdot\vec{n}\, du\, dv + \iint_{S_2} \vec{F}\cdot\vec{n}\, du\, dv$$

$$= \iint_{S_1} z^5\, du\, dv + \iint_{S_2} -z^5\, du\, dv$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (1-r^2)^{5/2}\, r\, dr\, d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} -\left(-(1-r^2)^{1/2}\right)^5 r\, dr\, d\theta$$

$$= 2\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (1-r^2)^{5/2}\, r\, dr\, d\theta$$

$$= 2\int_{\theta=0}^{\theta=2\pi} \frac{-1}{2}\,\frac{2}{7}(1-r^2)^{7/2}\Big|_{r=0}^{r=1} d\theta = \frac{-2}{7}\int_{\theta=0}^{\theta=2\pi}\left((1-1)^{7/2}-(1-0)^{7/2}\right)d\theta$$

$$= \frac{2}{7}\int_{\theta=0}^{\theta=2\pi} d\theta = \frac{2}{7}\theta\Big|_{\theta=0}^{\theta=2\pi} = \frac{2}{7}\cdot 2\pi = \frac{4\pi}{7}.$$

Solution by Gauss divergence theorem.

Let V be the sphere of radius 1.

(a) $\vec{F} = z^4\hat{k}$

Flux across $S = \iint_S \vec{F}\cdot\vec{n}\,du\,dv = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1}\int_{z=-\sqrt{1-\rho^2}}^{z=\sqrt{1-\rho^2}} 4z^3\rho\,dz\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1} z^4\Big|_{z=-\sqrt{1-\rho^2}}^{z=\sqrt{1-\rho^2}} \rho\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1} \left((1-\rho^2)^2 - \left(-(1-\rho^2)^{\frac{1}{2}}\right)^4\right)\rho\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1} 0\,\rho\,d\rho\,d\theta = 0.$$

(b) $\vec{F} = z^5\hat{k}$.

Flux across $S = \iint_S \vec{F}\cdot\vec{n}\,du\,dv = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1}\int_{z=-(1-\rho^2)^{\frac{1}{2}}}^{z=(1-\rho^2)^{\frac{1}{2}}} 5z^4\rho\,dz\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1} z^5\Big|_{z=-(1-\rho^2)^{\frac{1}{2}}}^{z=(1-\rho^2)^{\frac{1}{2}}} \rho\,d\rho\,d\theta$$

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$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} \left((1-\rho^2)^{5/2} - \left(-(1-\rho^2)^{\frac{1}{2}}\right)^5 \right) \rho \, d\rho \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} 2(1-\rho^2)^{5/2} \rho \, d\rho \, \theta$$

$$= \int_{\theta=0}^{\theta=2\pi} -\frac{2}{7}(1-\rho^2)^{7/2} \Big|_{\rho=0}^{\rho=1} d\theta$$

$$= -\frac{2}{7} \int_{\theta=0}^{\theta=2\pi} (0-1) \, d\theta = \frac{2}{7} \theta \Big|_{\theta=0}^{2\pi}$$

$$= \frac{2}{7} \cdot 2\pi = \frac{4}{7}\pi.$$