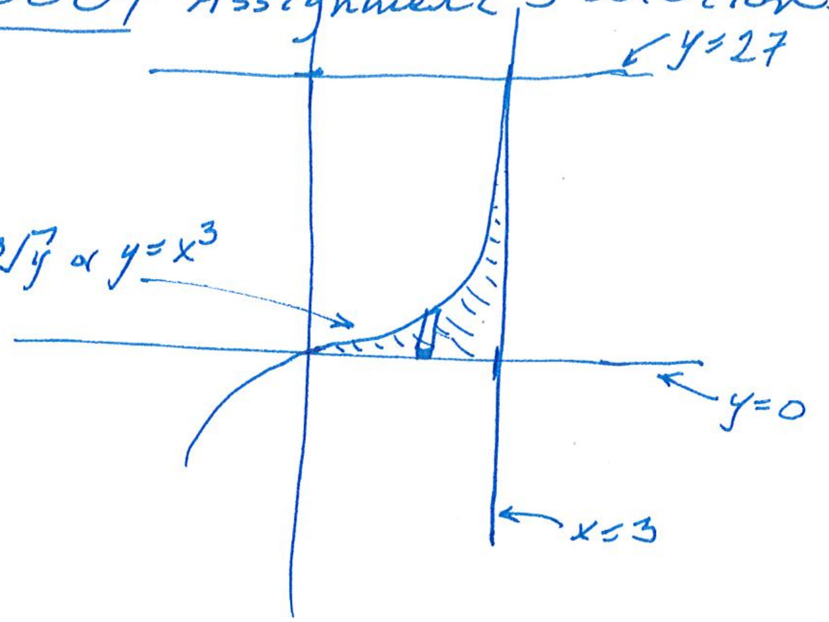
MAST 20009 Assignment 3 Solutions

(1)



$$\int_{y=0}^{y=27}\int_{x=\sqrt[3]{y}}^{x=3} \sinh(x^2)\,dx\,dy = \int_{x=0}^{x=3}\int_{y=0}^{y=x^3} \sinh(x^2)\,dy\,dx$$

$$= \int_{x=0}^{x=3}\left(y\sinh(x^2)\Big|_{y=0}^{y=x^3}\right)dx$$

$$= \int_{x=0}^{x=3} x^3\sinh(x^2)\,dx$$

$$= \frac{x^2}{2}\cosh(x^2) - \frac{1}{2}\sinh(x^2)\Big|_{x=0}^{x=3}$$ (see below for the integral)

$$= \frac{3^2}{2}\cosh(3^2) - \frac{1}{2}\sinh(3^2) - \left(\frac{0^2}{2}\cosh(0^2) - \frac{1}{2}\sinh(0^2)\right)$$

$$= \frac{9}{2}\cosh(9) - \frac{1}{2}\sinh(9) - (0-0) = \frac{9}{2}\cosh(9) - \frac{1}{2}\sinh(9).$$

The integral can be done as follows:

$$\int x^3 \sinh(x^2)\,dx = \int \frac{x^2}{2}\left(2x\sinh(x^2)\right)dx$$

$$= \frac{x^2}{2}\cosh(x^2) - \int x\cosh(x^2)\,dx$$

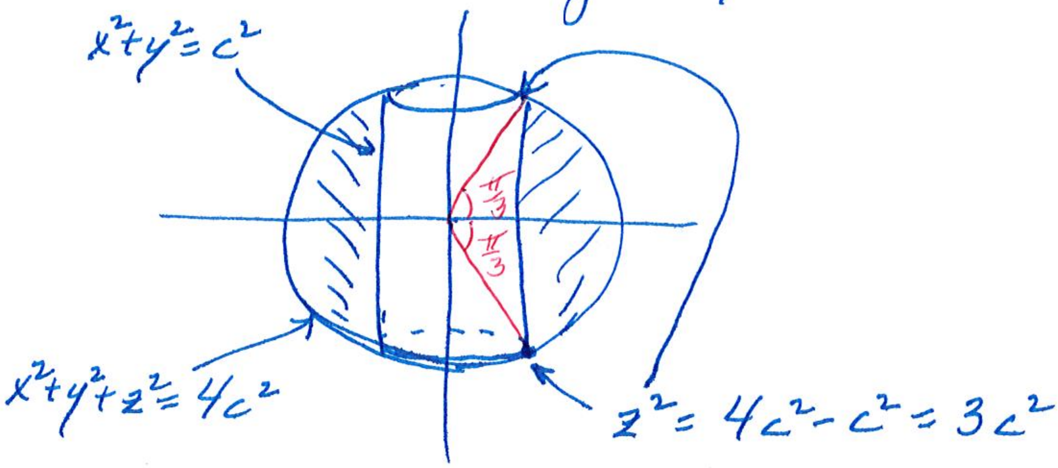
since $\frac{d}{dx}\left(\frac{x^2}{2}\cosh(x^2)\right) = \frac{x^2}{2}\left(2x\sinh(x^2)\right) + x\cosh(x^2)$.

So

$$\int x^3 \sinh(x^2)\,dx = \frac{x^2}{2}\cosh(x^2) - \int x\cosh(x^2)\,dx$$

$$= \frac{x^2}{2}\cosh(x^2) - \frac{1}{2}\sinh(x^2) + \text{constant}.$$

(2) To make the writing simpler let $c = 6352$.



The moment of inertia about the z-axis is

$$\iiint_D (x^2+y^2)\,dx\,dy\,dz$$

<u>Cylindrical coordinates</u>

$$x = \rho\cos\varphi \qquad dx\,dy\,dz = \rho\,d\rho\,d\varphi\,dz$$
$$y = \rho\sin\varphi$$
$$z = z$$

$$\iiint_D (x^2+y^2)\,dx\,dy\,dz = \iiint_D \rho^2\cdot\rho\,d\rho\,d\varphi\,dz$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\int_{z=-\sqrt{3}c}^{z=\sqrt{3}c}\int_{\rho=c}^{\rho=\sqrt{4c^2-z^2}} \rho^3\,d\rho\,dz\,d\varphi$$

$$\int_{\varphi=0}^{\varphi=2\pi}\int_{z=-\sqrt{3}c}^{z=\sqrt{3}c}\left(\frac{1}{4}\rho^4\Big|_{\rho=c}^{\rho=\sqrt{4c^2-z^2}}\right)dz\,d\varphi$$

$$=\int_{\varphi=0}^{\varphi=2\pi}\int_{z=-\sqrt{3}c}^{z=\sqrt{3}c}\frac{1}{4}\left(\left(4c^2-z^2\right)^2-c^4\right)dz\,d\varphi$$

$$=\int_{\varphi=0}^{\varphi=2\pi}\int_{z=-\sqrt{3}c}^{z=\sqrt{3}c}\frac{1}{4}\left(16c^4-8c^2z^2+z^4-c^4\right)dz\,d\varphi$$

$$=\int_{\varphi=0}^{\varphi=2\pi}\frac{1}{4}\left(15c^4z-\frac{8}{3}c^2z^3+\frac{z^5}{5}\right)\Big|_{z=-\sqrt{3}c}^{z=\sqrt{3}c}d\varphi$$

$$=\frac{1}{4}\int_{\varphi=0}^{\varphi=2\pi}\left(15c^4\left(\sqrt{3}c-\left(-\sqrt{3}c\right)\right)-\frac{8}{3}c^2\left(3\sqrt{3}c^3-\left(-3\sqrt{3}c^3\right)\right)+\left(\frac{9\sqrt{3}}{5}c^5-\left(-\frac{9\sqrt{3}}{5}c^5\right)\right)\right)d\varphi$$

$$=\frac{1}{4}\int_{\varphi=0}^{\varphi=2\pi}\left(30c^5\sqrt{3}-16c^5\sqrt{3}+\frac{18}{5}c^5\sqrt{3}\right)d\varphi$$

$$=\frac{1}{4}c^5\sqrt{3}\left(30-16+\frac{18}{5}\right)\varphi\Big|_{\varphi=0}^{\varphi=2\pi}=\frac{2\pi}{4}c^5\sqrt{3}\left(14+\frac{18}{5}\right)$$

$$=\frac{\pi}{2}c^5\sqrt{3}\left(\frac{70+18}{5}\right)=\frac{\pi}{2}c^5\sqrt{3}\cdot\frac{88}{5}=\frac{44}{5}c^5\sqrt{3}\pi$$

$$=\frac{44}{5}\cdot 635\,2^5\cdot\sqrt{3}\pi.$$

Spherical coordinates

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$ $\qquad dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\varphi$

$z = r\cos\theta$

Since $x^2+y^2=c^2$ is $r^2\sin^2\theta = c^2$ then

$$\iiint_D (x^2+y^2)\,dx\,dy\,dz = \iiint_D r^2\sin^2\theta\, r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=\pi/6}^{\theta=\frac{5\pi}{6}}\int_{r=\frac{c}{\sin\theta}}^{r=2c} r^4\sin^3\theta\,dr\,d\theta\,d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=\pi/6}^{\theta=5\pi/6}\left(\frac{r^5}{5}\sin^3\theta\Bigg|_{r=\frac{c}{\sin\theta}}^{r=2c}\right)d\theta\,d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=\pi/6}^{\theta=5\pi/6}\left(\frac{32c^5}{5} - \frac{c^5}{5\sin^5\theta}\right)\sin^3\theta\,d\theta\,d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=-\pi/6}^{\theta=\pi/6}\left(\frac{32}{5}c^5\sin^3\theta - \frac{c^5}{5}\,\frac{1}{\sin^2\theta}\right)d\theta\,d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\left(\frac{32}{5}c^5\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right) + \frac{c^5}{5}\,\frac{\cos\theta}{\sin\theta}\right)\Bigg|_{\theta=\pi/6}^{\theta=\frac{5\pi}{6}}d\varphi$$

(see below for explanation of $\int\sin^3\theta\,d\theta$ and $\int\frac{1}{\sin^2\theta}\,d\theta$)

$$= \int_{\varphi=0}^{\varphi=2\pi} \left(\frac{32}{5} c^5 \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{3}\left(\frac{-3\sqrt{3}}{8} - \frac{3\sqrt{3}}{8} \right) \right) + \frac{c^5}{5}\left(-\sqrt{3} - \sqrt{3} \right) \right) d\varphi$$

$$= \left(\frac{32}{5} c^5 \left(\sqrt{3} - \frac{2\sqrt{3}}{8} \right) + \frac{c^5}{5}\left(-2\sqrt{3} \right) \right) \varphi \Big|_{\varphi=0}^{\varphi=2\pi}$$

$$= \frac{c^5}{5}\sqrt{3}\left(32\left(1 - \frac{1}{4}\right) - 2 \right) 2\pi = \frac{c^5}{5}\sqrt{3}\left(32 \cdot \frac{3}{4} - 2 \right) 2\pi$$

$$= \frac{c^5}{5}\sqrt{3}\,(24-2)\cdot 2\pi = \frac{c^5}{5}\sqrt{3}\cdot 22 \cdot 2\pi = \frac{44}{5}\pi c^5 \cdot \sqrt{3}$$

$$= \frac{44}{5} \cdot 6352^5 \cdot \pi\sqrt{3}.$$

The integrals:

$$\int \sin^3\theta \, d\theta = \int \sin\theta \sin^2\theta \, d\theta = \int \sin\theta (1 - \cos^2\theta) \, d\theta$$

$$= -\cos\theta + \frac{\cos^3\theta}{3} + \text{constant}$$

and since

$$\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta} \right) = \frac{d}{d\theta}\left(\cos\theta (\sin\theta)^{-1} \right)$$

$$= -\sin\theta (\sin\theta)^{-1} + \cos\theta(-1)(\sin\theta)^{-2}\cos\theta$$

$$= -\frac{\sin\theta}{\sin\theta} - \frac{\cos^2\theta}{\sin^2\theta} = -1 - \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} = \frac{-1}{\sin^2\theta}$$

then

$$\int \frac{-1}{\sin^2\theta} \, d\theta = \frac{\cos\theta}{\sin\theta} + \text{constant}.$$

(3) $\vec{F}(x,y,z) = e^{y+2z}(\hat{\imath} + x\hat{\jmath} + 2x\hat{k})$

(a) The straight line from $(0,0,0)$ to $(1,1,1)$ along $x=y=z$ has $c(t) = (t,t,t) = (x,y,z)$

so $\vec{F} = e^{t+2t}(1,t,2t)$, $\frac{dc}{dt} = (1,1,1)$

and $\vec{F}\cdot\frac{dc}{dt} = e^{3t}(1+t+2t) = e^{3t} + 3te^{3t}$

So the work done is

$$\int_{t=0}^{t=1} \vec{F}\cdot\frac{dc}{dt}\,dt = \int_{t=0}^{t=1} (e^{3t} + 3te^{3t})\,dt$$

$$= te^{3t}\Big|_{t=0}^{t=1} = 1\cdot e^{3\cdot 1} - 0\cdot e^{3\cdot 0} = e^3.$$

(b) The curve $c(t) = (t, t^2, t^4) = (x,y,z)$ has

$\vec{F} = e^{t^2+2t^4}(1,t,2t)$, $\frac{dc}{dt} = (1, 2t, 4t^3)$

and $\vec{F}\cdot\frac{dc}{dt} = e^{t^2+2t^4}(1+2t^2+8t^4)$

$$= e^{t^2+2t^4} + t(2t+8t^3)e^{t^2+2t^4}.$$

So the work done is

$$\int_{t=0}^{t=1} \vec{F}\cdot\frac{dc}{dt}\,dt = \int_{t=0}^{t=1} \left(e^{t^2+2t^4} + t(2t+8t^3)e^{t^2+2t^4}\right)dt$$

$$= te^{t^2+2t^4}\Big|_{t=0}^{t=1} = 1e^{1^2+2\cdot 1^4} - 0\cdot e^{0^2+2\cdot 0^4} = e^3.$$

(c) The curve is the combination of

$c_1(t) = (t,0,0)$ with $\frac{dc_1}{dt} = (1,0,0)$,

$\vec{F} = e^{0+2\cdot 0}(1,t,2t)$ and $\vec{F}\cdot\frac{dc_1}{dt} = 1+0+0 = 1$

and

$c_2(t) = (1,t,0)$ with $\frac{dc_2}{dt} = (0,1,0)$

$\vec{F} = e^{t+2\cdot 0}(1,1,2)$ and $\vec{F}\cdot\frac{dc_2}{dt} = e^t(0+1+0) = e^t$

and

$c_3(t) = (1,1,t)$ with $\frac{dc_3}{dt} = (0,0,1)$,

$\vec{F} = e^{1+2t}(1,1,2)$ and $\vec{F}\cdot\frac{dc_3}{dt} = e^{1+2t}(0+0+2)$

So the total work done is

$$\int_0^1 1\cdot dt + \int_0^1 e^t dt + \int_0^1 2e^{1+2t}dt$$

$$= (t + e^t + e^{1+2t})\Big|_{t=0}^{t=1} = 1+e^1+e^{1+2} - (0+e^0+e^{1+2\cdot 0})$$

$$= 1+e+e^3 - (0+1+e) = e^3.$$

The force is conservative since

if $f(x,y,z) = xe^{y+2z}$ then

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(e^{y+2z}, xe^{y+2z}, 2xe^{y+2z}\right)$$

$$= \vec{F}(x,y,z).$$

Indeed

$$f(1,1,1) - f(0,0,0) = 1 \cdot e^{1+2\cdot 1} - 0 \cdot e^{0+2\cdot 0}$$

$$= e^3 - 0 = e^3.$$