MAST20009 Semester 2, 2017
Assignment 1, Solutions

(1) $f(x,y) = \dfrac{x^2-y^2}{x^3-y^3} = \dfrac{(x-y)(x+y)}{(x-y)(x^2+xy+y^2)}$

$$= \frac{x+y}{x^2+xy+y^2}$$

(a) Since

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x) = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x+y}{x^2+xy+y^2} = \lim_{x\to 0} \frac{x+0}{x^2+0+0}$$

$$= \lim_{x\to 0} \frac{1}{x} \quad \text{does not exist}$$

then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(b) $\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(x,y)\to(1,1)} \dfrac{x+y}{x^2+xy+y^2} = \dfrac{1+1}{1+1+1} = \dfrac{2}{3}.$

(c) Since

$x+y$ is continuous at $(1,1)$ (it is a polynomial)

x^2+xy+y^2 is continuous at $(1,1)$,

and x^2+xy+y^2 is not 0 at $(1,1)$

then

$f(x,y) = \dfrac{x+y}{x^2+xy+y^2}$ is continuous at $(1,1)$.

The function $f: \{(x,y)\in\mathbb{R}^2 \mid x\neq y\} \to \mathbb{R}$ given by $f(x,y) = \dfrac{x^2-y^2}{x^3-y^3}$ is not continuous at $(1,1)$ since it is not defined at $(1,1)$.

(a) Let $x = \frac{r\cos\theta}{\sqrt{3}}$ and $y = \frac{r\sin\theta}{2}$.

Then, when $(x,y) \neq (0,0)$

$$g(x,y) = \frac{x^2 y}{3x^2+4y^2} = \frac{\frac{r^2\cos^2\theta}{3} \cdot \frac{r\sin\theta}{2}}{3 \cdot \frac{r^2\cos^2\theta}{3} + 4r^2\frac{\sin^2\theta}{4}}$$

$$= \frac{\frac{1}{6}r^3\cos^2\theta\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)} = \frac{1}{6}r\cos^2\theta\sin\theta$$

Since $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$ then

$$|g(x,y)| = \left|\tfrac{1}{6}r\cos^2\theta\sin\theta\right| \le \tfrac{1}{6}r.$$

So

$$0 \le \lim_{(x,y)\to(0,0)} |g(x,y)| \le \lim_{r\to 0} \tfrac{1}{6}r = 0.$$

So $\displaystyle\lim_{(x,y)\to(0,0)} |g(x,y)| = 0$.

So $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y) = 0$.

(b) Since $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$

then $g(x,y)$ is continuous at $(0,0)$.

Since x^2y is a polynomial (hence continuous) and $3x^2+y^2$ is a polynomial and $3x^2+y^2 \neq 0$ when $(x,y)\neq(0,0)$

then $g(x,y)$ is continuous when $(x,y)\neq(0,0)$

(c) $$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(x^2y(3x^2+4y^2)^{-1}\right)$$

$$= 2xy(3x^2+4y^2)^{-1} + x^2y(-1)(3x^2+4y^2)^{-2}\cdot 6x$$

$$= \frac{2xy}{3x^2+4y^2} + \frac{-x^2y\cdot 6x}{(3x^2+4y^2)^2}$$

$$= \frac{2xy(3x^2+4y^2) - x^2y\cdot 6x}{(3x^2+4y^2)^2}$$

$$= \frac{6x^3y + 8xy^3 - 6x^3y}{(3x^2+4y^2)^2} = \frac{8xy^3}{(3x^2+4y^2)^2}$$

Since $g(x,0) = \begin{cases} \dfrac{x^2\cdot 0}{3x^2+0^2} & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases} = 0$

then $\dfrac{\partial g}{\partial x} = 0$ when $(x,y)=(0,0)$

So $$\frac{\partial g}{\partial x} = \begin{cases} \dfrac{8xy^3}{(3x^2+4y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0) \end{cases}$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(x^2y\,(3x^2+4y^2)^{-1}\right)$$

$$= \cancel{2xy}\; x^2(3x^2+4y^2)^{-1} + x^2y(-1)(3x^2+4y^2)^{-2}\cdot 8y$$

$$= \frac{x^2}{3x^2+4y^2} + \frac{-x^2y\cdot 8y}{(3x^2+4y^2)^2}$$

$$= \frac{x^2(3x^2+4y^2) - 8x^2y^2}{(3x^2+4y^2)^2} = \frac{3x^4+4x^2y^2-8x^2y^2}{(3x^2+4y^2)^2}$$

$$= \frac{3x^4-4x^2y^2}{(3x^2+4y^2)^2} = \frac{x^2(3x^2-4y^2)}{(3x^2+4y^2)^2}$$

Since $g(0,y) = \left\{\begin{array}{ll} \dfrac{0^2\cdot y}{0+4y^2}, & \text{if } y\neq 0 \\ 0, & \text{if } y=0 \end{array}\right\} = 0$

then $\frac{\partial g}{\partial y} = 0$ when $(x,y)=(0,0)$

So $\frac{\partial g}{\partial y} = \begin{cases} \dfrac{x^2(3x^2-4y^2)}{(3x^2+4y^2)^2}, & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$

(d) $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=y}} \dfrac{\partial g}{\partial x} = \lim\limits_{\substack{(x,y)\to(0,0)\\ x=y}} \dfrac{8xy^3}{(3x^2+4y^2)^2}$

$= \lim\limits_{x\to 0} \dfrac{8x\cdot x^3}{(3x^2+4x^2)^2} = \lim\limits_{x\to 0} \dfrac{8x^4}{(7x^2)^2}$

$= \lim\limits_{x\to 0} \dfrac{8}{49} = \dfrac{8}{49}.$

Since $\frac{8}{49}$ is not equal to $\frac{\partial g}{\partial x}$ at $(0,0)$

(recall that $\frac{\partial g}{\partial x}$ is 0 at $(x,y)=(0,0)$)

then $\frac{\partial g}{\partial x}$ is not continuous at $(0,0)$.

So $g(x,y)$ is not C^1.

(3) $h(x,y) = \sqrt{35-6x^2-2y^2}$

(a) Since $h(2,-1) = \sqrt{35-6\cdot 2^2-2(-1)^2}$

$$= \sqrt{35-24-2} = \sqrt{35-26} = \sqrt{9} = 3$$

and $(2.02,-0.96)$ is close to $(2,-1)$

then $(2,-1)$ is an appropriate point about which to approximate $h(x,y)$.

(b) $$h(x,y) \simeq h(2,1) + (x-2)\left.\frac{\partial h}{\partial x}\right|_{(2,-1)} + (y-(-1))\left.\frac{\partial h}{\partial y}\right|_{(2,-1)}$$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left((35-6x^2-2y^2)^{\frac12}\right) = \frac12(35-6x^2-2y^2)^{-\frac12}(-12x)$$

$$= \frac{-6x}{(35-6x^2-2y^2)^{1/2}}$$ and

$$\left.\frac{\partial h}{\partial x}\right|_{(2,-1)} = \frac{-6\cdot 2}{\sqrt{35-6\cdot 2^2-2(-1)^2}} = \frac{-12}{3} = -4.$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left((35-6x^2-2y^2)^{\frac12}\right) = \frac12(35-6x^2-2y^2)^{-\frac12}(-4y)$$

$$= \frac{-2y}{(35-6x^2-2y^2)^{1/2}}$$ and

$$\left.\frac{\partial h}{\partial y}\right|_{(2,-1)} = \frac{-2(-1)}{(35-6\cdot 2^2-2(-1)^2)^{\frac12}} = \frac{2}{3} = \frac{2}{3}.$$ So

$$h(2.02,-0.96) \simeq 3 + (.02)(-4) + (.04)\cdot\frac{2}{3} = 3 - .08 + \frac{.08}{3}$$
$$= 3 - \frac{.16}{3} \simeq 3 - .05 = 2.95$$