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A SHORT ELEMENTARY PROOF OF GROTHENDIECK'S THEOREM ON ALGEBRAIC VECTORBUNDLES OVER THE PROJECTIVE LINE*

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Let E be an algebraic (or holomorphic) vectorbundle over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. Then Grothendieck proved that E splits into a sum of line bundles $E=\bigoplus L_i$ and that the isomorphism classes of the L_i are (up to order) uniquely determined by E. The L_i in turn are classified by an integer (their Chern numbers) so that m-dimensional vectorbundles over $\mathbb{P}^1(\mathbb{C})$ are classified by an m-tuple of integers

$$\kappa(E)=(\kappa_1(E),\dots,\kappa_m(E)),\quad \kappa_1(E)\geq\kappa_2(E)\geq\cdots\geq\kappa_m(E),\quad \kappa_i(E)\in\mathbb{Z}.$$

In this short note we present a completely elementary proof of these facts which, as it turns out, works over any field k.

1. Introduction

Let E be a holomorphic (or algebraic) vectorbundle over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. (By [2] holomorphic and algebraic vectorbundles over $\mathbb{P}^1(\mathbb{C})$ amount to the same thing). In [1] Grothendieck proved that E splits into a sum of line bundles $E=\bigoplus L_i$ and that the isomorphism classes of the L_i are (up to order) uniquely determined by E. The line bundles L_i in turn are classified by an integer (their first Chern number) so that m-dimensional vectorbundles over $\mathbb{P}^1(\mathbb{C})$ are classified by an m-tuple of integers

$$\kappa(E)=(\kappa_1(E),\dots,\kappa_m(E)),\quad \kappa_1(E)\geq\cdots\geq\kappa_m(E),\quad \kappa_i\in\mathbb{Z}.$$

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Below we give a completely elementary proof of these facts, which, as it turns out, works over any field k. Of course 'completely elementary' means that such concepts as 'degree of a line bundle' or 'first Chern number' or 'cohomology' or 'intersection number' are not needed or mentioned below. All we use is some linear algebra (or matrix manipulation).

2. Vectorbundles over $\mathbb{P}^1_k$

Let k be any field. The projective line $\mathbb{P}^1_k$ over k can be obtained as follows. Let $U_1 = \mathrm{Spec}(k[s])$, $U_2 = \mathrm{Spec}(k[t])$, $U_{12} = \mathrm{Spec}(k[s,s^{-1}]) = U_1 \setminus \{0\}$, $U_{21} = \mathrm{Spec}(k[t,t^{-1}]) = U_2 \setminus \{0\}$. Now glue U_1 and U_2 together by identifying U_{12} and U_{21} by means of the isomorphism

$$k[s,s^{-1}] \xrightarrow{\sim} k[t,t^{-1}], \quad s \mapsto t^{-1}.$$

Now let E be an m-dimensional vectorbundle over $\mathbb{P}^1_k$ defined over k_1 and let $\mathbb{A}^m = \mathrm{Spec}(k[X_1, \ldots, X_m])$. Then $E|_{U_i}$, $i=1,2$, is trivial, i.e. $E|_{U_i} \simeq U_i \times \mathbb{A}^m$, so that E can be viewed (up to isomorphism) as obtained by glueing together $U_1 \times \mathbb{A}^m$ and $U_2 \times \mathbb{A}^m$ by identifying $U_1 \setminus \{0\} \times \mathbb{A}^m$ and $U_2 \setminus \{0\} \times \mathbb{A}^m$ by means of an isomorphism of the form

$$(s, v) \mapsto (s^{-1}, A(s,s^{-1})v) \tag{2.1}$$

where $A(s,s^{-1})$ is a matrix with coefficients in $k[s,s^{-1}]$ which has nonzero determinant for all $s \neq 0$, $s^{-1} \neq 0$. This last fact means that

$$\det(A(s,s^{-1})) = s^n, \quad n \in \mathbb{Z}. \tag{2.2}$$

A vectorbundle automorphism of $U_1 \times \mathbb{A}^m$ is necessarily of the form $(s,v) \to (s, U(s)v)$ where $U(s)$ is a matrix with coefficients in $k[s]$ with $\det U(s) \in k \setminus \{0\}$ and similarly an automorphism of $U_2 \times \mathbb{A}^m$ is given by a matrix $V(s^{-1})$ with coefficients in $k[s^{-1}]$ with determinant in $k \setminus \{0\}$. Different trivializations of $E|_{U_i}$ differ by an automorphism of $U_i \times \mathbb{A}^m$. It follows that

Proposition 2.3. *Isomorphism classes of m-dimensional algebraic vectorbundles over $\mathbb{P}^1_k$ correspond bijectively to equivalence classes of polynomial $m \times m$ matrices $A(s,s^{-1})$ over $k[s,s^{-1}]$ such that $\det A(s,s^{-1}) = s^n$, $n \in \mathbb{Z}$ where the equivalence relation is the following: $A(s,s^{-1}) \sim A'(s,s^{-1})$ iff there exist polynomial invertible $m \times m$ matrices $U(s)$, $V(s^{-1})$ over $k[s]$ and $k[s^{-1}]$ respectively with constant determinant such that*

$$A'(s,s^{-1}) = V(s^{-1})A(s,s^{-1})U(s). \tag{2.4}$$

3. A canonical form for matrices over $k[s, s^{-1}]$

Now let us study canonical forms for $m\times m$ matrices over $k[s,s^{-1}]$ under the equivalence relation defined in Proposition 2.3 above. The result is

Proposition 3.1. *Let $A(s,s^{-1})$ be an $m\times m$ matrix over $k[s,s^{-1}]$ with determinant equal to s^n for some $n\in\mathbb{Z}$. Then there exist polynomial $m\times m$ matrices $V(s^{-1})$ and $U(s)$ with constant nonzero determinant such that*

$$V(s^{-1})A(s,s^{-1})U(s)=\begin{pmatrix} s^{r_1} & & & 0 \\ & s^{r_2} & & \\ & & \ddots & \\ 0 & & & s^{r_m}\end{pmatrix} \tag{3.2}$$

with $r_1\geq r_2\geq\cdots\geq r_m$, $r_i\in\mathbb{Z}$. The r_i are uniquely determined by $A(s,s^{-1})$. Moreover if $A(s,s^{-1})$ is polynomial in s then $r_i\geq 0$, $i=1,\ldots,m$, and if $A(s,s^{-1})$ is polynomial in s^{-1} then $r_i\leq 0$, $i=1,\ldots,m$.

Proof. Let's prove uniqueness first. Write $D(r_1,\ldots r_m)$ for the matrix on the right in (3.2). Suppose there were two such matrices equivalent to $A(s,s^{-1})$. Then there would be polynomial matrices with constant nonzero determinant $U(s)$, $V(s^{-1})$ such that

$$V(s^{-1})D(r_1,\ldots,r_m)=D(r_1',\ldots,r_m')U(s).$$

If A is a matrix let

$$A^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$$

denote the minor of A obtained by taking the determinant of the submatrix of A obtained by removing all rows with index in $\{1,\ldots,m\}\backslash\{i_1,\ldots,i_k\}$ and all columns with index in $\{1,\ldots,m\}\backslash\{j_1,\ldots,j_k\}$. Then of course

$$(AB)^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}=\sum_{r_1<\cdots<r_k}A^{i_1,\ldots,i_k}_{r_1,\ldots,r_k}B^{r_1,\ldots,r_k}_{j_1,\ldots,j_k}.$$

Using this on the equality $V(s^{-1})D(r_1,\ldots,r_m)=D(r_1',\ldots,r_m')U(s)$ one finds that

$$V^{1,2,\ldots,k}_{i_1,\ldots,i_k}(s^{-1})s^{r_{i_1}+\cdots+r_{i_k}}=s^{r_1'+\cdots+r_k'}U^{1,2,\ldots,k}_{i_1,\ldots,i_k}(s) \tag{3.3}$$

for all $i_1<\cdots<i_k$. Now for some $i_1,\ldots,i_k$,

$$U^{1,2,\ldots,k}_{i_1,\ldots,i_k}(s)\neq 0.$$

Hence $r_1'+\cdots+r_k'\leq r_{i_1}+\cdots+r_{i_k}$ for some $i_1<\cdots<i_k$, and hence certainly $r_1'+\cdots+r_k'\leq r_1+\cdots+r_k$ for all k. Multiplying with $V(s^{-1})^{-1}$ on the left and $U(s)^{-1}$ on the right in $V(s^{-1})D(r_1,\ldots,r_m)=D(r_1',\ldots,r_m')U(s)$ and repeating the argument gives $r_1+\cdots+r_k\leq r_1'+\cdots r_k'$ for all k and hence $r_i=r_i'$, $i=1,\ldots,m$.

It remains to prove existence. First multiply $A(s,s^{-1})$ with a suitable power s^n, $n\in\mathbb{N}\cup\{0\}$ to obtain a polynomial matrix $B(s)$. Then by post multiplication with a suitable $U(s)$ (column operations) we can find a $B'(s)$ with $b'_{11}\neq 0$ and $b'_{1i}=0$, $i=2,\dots,m$ (b'_{11} is the greatest common divisor of $b_{11},\dots,b_{1m}$). Of course $b'_{11}=s^{k_1}$ for some $k_1\in\mathbb{N}\cup\{0\}$ because $\det B(s)$ is a power of s. Let B_2 be the lower-right $(m-1)\times(m-1)$ submatrix of B. By induction we can assume that the proposition holds for $(m-1)\times(m-1)$ matrices. (The case $m=1$ is trivial). So there are $U_2(s)$, $V_2(s^{-1})$ such that $V_2(s^{-1})B_2U_2(s)$ is of the form of the right hand side of (3.2). Then

$$C(s)=\begin{pmatrix}1 & 0\\ 0 & V_2\end{pmatrix}B\begin{pmatrix}1 & 0\\ 0 & U_2\end{pmatrix}=\begin{bmatrix} s^{k_1} & 0 & \cdots & 0\\ c_2 & s^{k_2} & & 0\\ \vdots & & \ddots & \\ \vdots & 0 & & \\ c_m & & & s^{k_m}\end{bmatrix} \tag{3.4}$$

for certain $k_1,k_2,\dots,k_m\in\mathbb{N}\cup\{0\}$ (same k_1 as before) and $c_i\in k[s,s^{-1}]$, $i=2,\dots,m$. Subtracting suitable $k[s^{-1}]$ multiplies of the first row from rows $2,\dots,m$ (which is premultiplication with a $V(s^{-1})$) we can moreover see to it that $c_i\in k[s]$.

Now consider all polynomial matrices of the form (3.4) which are equivalent to $B(s)$. Choose one for which k_1 is maximal. Such a one exist because $k_1\leq$ degree $(\det B(s))$ because $k_2,\dots,k_m\geq 0$. We claim that then $k_1\geq k_i$, $i=2,\dots,m$. Indeed suppose that $k_1<k_i$. Subtracting a suitable $k[s^{-1}]$ multiple of the first row from the i-th row we find a matrix (3.4) with $c_i=s^{k_1+1}c'(s)$. Now interchange the first and the i-th row to find a polynomial matrix $B'(s)$ such that the greatest common divisor of its first row elements is $s^{k'_1}$ with $k'_1\geq k_1+1$. Now apply to $B'(s)$ the same procedure as above to $B(s)$. This would give a $C'(s)$ of the form (3.4) with $k'_1>k_1$, a contradiction. We can therefore assume that in (3.4) $k_1\geq k_i$, $c_i\in k[s]$, $i=2,\dots,m$. Subtracting suitable $k[s]$-multiples of the 2-nd, ..., m-th columns from the first one we find a matrix (3.4) with degree $(c_i)\leq k_i$. But then $\deg(c_i)<k_1$ so that a suitable $k[s^{-1}]$ multiple of s^{k_1} is equal to c_i so that a further premultiplication with a $V(s^{-1})$ gives us a matrix (3.4) with $c_2=\cdots=c_m=0$. This proves the first half of the last part of the statement of the proposition and shows that there are $k_1,\dots,k_n\in\mathbb{N}\cup\{0\}$, $k_1\geq\cdots\geq k_m$ (by permuting columns and rows if necessary) and $U(s)$, $V(s^{-1})$ of constant nonzero determinant such that

$$V(s^{-1})s^nA(s,s^{-1})U(s)=V(s^{-1})B(s)U(s)=D(k_1,\dots,k_m).$$

Multiplying with s^{-n} gives $V(s^{-1})A(s,s^{-1})U(s)=D(r_1,\dots,r_m)$ with $r_i=k_i-n$. The second half of the last statement of the proposition is proved as the first half starting with a matrix $B(s^{-1})$ and using row (resp. column) operations everywhere where we used column (resp. row) operations above. This concludes the proof of Proposition 3.1.

4. Classification of vectorbundles over $\mathbb{P}^1_k$

Let $O(n)$, $n \in \mathbb{Z}$ be the line bundle over $\mathbb{P}^1_k$ defined by the glueing matrix $A(s,s^{-1}) = s^{-n}$. Obviously then the bundle defined by the glueing matrix $A(s,s^{-1}) = D(r_1,\dots,r_m)$ is equal to the direct sum $O(-r_1) \oplus \cdots \oplus O(-r_m)$.

Theorem 4.1. *Let E be an algebraic m-dimensional vectorbundle over $\mathbb{P}^1_k$ which is defined over k. Then E is isomorphic over k to a direct sum of line bundles*

$$E \simeq O(\kappa_1) \oplus \cdots \oplus O(\kappa_m), \quad \kappa_1 \geq \cdots \geq \kappa_m, \quad \kappa_i \in \mathbb{Z}.\ i = 1,\dots,m,$$

and the κ_i are uniquely determined by the isomorphism class of E.

Remarks 4.2. It is perhaps worth remarking that E is positive (meaning that all the $\kappa_i(E) \geq 0$) if the glueing matrix $A(s,s^{-1})$ is polynomial in s^{-1} and that E is negative (i.e. $\kappa_i(E) \leq 0$ all i) if $A(s,s^{-1})$ is polynomial in s. This follows from the last statement of Proposition 3.1. Also E contains a summand $O(n)$ with $n > 0$ if $\deg(\det A(S,s^{-1})) < 0$. Finally it follows that vectorbundles over $\mathbb{P}^1_k$ have no forms, i.e. if E and E' are two vectorbundles over k which become isomorphic over the algebraic closure $\bar{k}$ of k then E and E' are also isomorphic over k. This can of course also be seen by other, more sophisticated, means (e.g. Galois cohomology).

References

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