What is an abelian variety?

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Let $\omega_1, \ldots, \omega_{2g} \in \mathbb{C}^g$ and let

$$\Lambda = \mathbb{Z}\text{-span}\{\omega_1, \ldots, \omega_{2g}\}.$$

Let $\mathbb{C}^g/\Lambda$ have the quotient topology for $\mathbb{C}^g \to \mathbb{C}^g/\Lambda$.

A <u>complex torus</u> is $\mathbb{C}^g/\Lambda$ such that $\mathbb{C}^g/\Lambda$ is a complex manifold.

An <u>abelian variety</u> is a complex torus $\mathbb{C}^g/\Lambda$ which embeds into projective space.

An <u>elliptic curve</u> is an abelian variety $\mathbb{C}^g/\Lambda$ with $g=1$.

A <u>polarized abelian variety</u> is a pair $(\mathbb{C}^g/\Lambda, \mathcal{L})$ where $\mathbb{C}^g/\Lambda$ is an abelian variety and $\mathcal{L}$ is an ample line bundle on $\mathbb{C}^g/\Lambda$.

<u>Harder Example 16 p. 42</u> Let

$$\Lambda = \{n_1 \omega_1 + n_2 \omega_2 \mid n_1, n_2 \in \mathbb{Z}\} = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\}$$

with ω_1, ω_2 linearly independent over $\mathbb{R}$.

$\mathbb{C}/\Lambda$ has the quotient topology for $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Lambda$. Define $\mathcal{O}_{\mathbb{C}/\Lambda}$ by

$$\mathcal{O}_{\mathbb{C}/\Lambda}(U) = \left\{ f: U \to \mathbb{C} \;\middle|\; \pi^{-1}(U) \xrightarrow{f \circ \pi} \mathbb{C} \text{ is holomorphic} \right\}$$

Then $\mathbb{C}/\Omega$ is a complex manifold.

<u>Harder §5.1.6</u>

A <u>compact Riemann surface</u> S is a compact complex manifold of dimension 1.

Then

$$H^0(S, \mathbb{Z}) = \mathbb{Z}, \quad H^1(S, \mathbb{Z}) = \mathbb{Z}^{2g}, \quad H^2(S, \mathbb{Z}) = \mathbb{Z}$$

and $H^i(S, \mathbb{Z}) = 0$ for $i \in \mathbb{Z}_{>2}$.

The <u>genus</u> of S is

$$g = \tfrac{1}{2} \operatorname{rank}(H^1(S, \mathbb{Z})).$$

<u>Proposition</u> Let S be a compact Riemann surface and let g be the genus of S.

(a) If $g=0$ then $S \simeq \mathbb{P}^1(\mathbb{C})$.

(b) If $g=1$ then $S \simeq \mathbb{C}/\Lambda$

for some $\Lambda = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\}$ with $\omega_1, \omega_2 \in \mathbb{C}$ which are $\mathbb{R}$-linearly independent.

<u>Idea of proof of (b):</u>

Let $s_0 \in S$ and

ω a generator of $\Omega^1_{S/\mathbb{C}}$.

Let $\tilde{S} = \left\{ (s,\gamma) \;\middle|\; \begin{array}{l} s\in S,\ \gamma \text{ is a homotopy class of} \\ \text{a path from } s_0 \text{ to } s \end{array} \right\}$

and define

$$\begin{aligned} h: \tilde{S} &\longrightarrow \mathbb{C} \\ (s,\gamma) &\longmapsto \int_\gamma \omega \end{aligned}$$

Then

$$\begin{tikzcd} \tilde{S} \arrow[r, "\sim"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi"] \\ S \arrow[r, "\sim"] & \mathbb{C}/\Lambda \end{tikzcd} \qquad \begin{tikzcd} (s,\gamma) \arrow[r, mapsto] \arrow[d, mapsto] & \int_\gamma \omega \\ s & \end{tikzcd}$$

and $S \simeq \mathbb{C}/\Lambda$ where $\Lambda = (h\circ\pi^{-1})(s_0)$

Indexing complex tori

Let $\omega_1, \dots, \omega_{2g} \in \mathbb{C}^g$,

$$\Omega = \begin{pmatrix} \text{---}\ \omega_1\ \text{---} \\ \text{---}\ \omega_2\ \text{---} \\ \vdots \\ \text{---}\ \omega_{2g}\ \text{---} \end{pmatrix} \in M_{2g\times g}(\mathbb{C}), \quad \text{and}$$

$$\Lambda_\Omega = \mathbb{Z}\text{-span}\{\omega_1, \dots, \omega_{2g}\}.$$

Proposition $\mathbb{C}^g/\Lambda_\Omega$ is a compact complex manifold

$\Leftrightarrow$ $\mathrm{rank}(\Lambda_\Omega) = 2g$ $\Leftrightarrow$ $\det(\Omega, \overline{\Omega}) \neq 0$.

If $M \in GL_{2g}(\mathbb{Z})$ then $\Lambda_\Omega = \Lambda_{M\Omega}$

If $k \in GL_g(\mathbb{C})$ then $\mathbb{C}^g/\Lambda_\Omega \simeq \mathbb{C}^g/\Lambda_{\Omega k}$

Theorem Let

$$\mathcal{M} = \{\Omega \in M_{2g\times g}(\mathbb{C}) \mid \det(\Omega, \overline{\Omega}) \neq 0\}$$

Then

$$GL_{2g}(\mathbb{Z}) \backslash \mathcal{M} / GL_g(\mathbb{C}) \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \text{isomorphism classes} \\ \text{of complex tori} \end{matrix} \right\}$$

$$\Omega \longmapsto \mathbb{C}^g/\Lambda_\Omega$$

The action $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau = (A\tau + B)(C\tau+D)^{-1}$

Let

$$\Omega = \begin{pmatrix} \text{---}\ \omega_1\ \text{---} \\ \text{---}\ \omega_2\ \text{---} \\ \vdots \\ \text{---}\ \omega_{2g}\ \text{---} \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in M_{2g\times g}(\mathbb{C}).$$

Let $\sigma \in S_{2g}$ (the symmetric group $S_{2g} \subseteq GL_{2g}(\mathbb{Z})$)

so that

$$\sigma\Omega = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \quad \text{with } \det(\tau_2) \neq 0$$

Then

$$\sigma\Omega\tau_2^{-1} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \tau_2^{-1} = \begin{pmatrix} \tau_1\tau_2^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2g}(\mathbb{Z})$ then

$$M\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} A\tau + B \\ C\tau + D \end{pmatrix}$$

and

$$M\begin{pmatrix} \tau \\ 1 \end{pmatrix}(C\tau+D)^{-1} = \begin{pmatrix} A\tau + B \\ C\tau + D \end{pmatrix}(C\tau+D)^{-1} = \begin{pmatrix} (A\tau+B)(C\tau+D)^{-1} \\ 1 \end{pmatrix}$$

Indexing abelian varieties

The Siegel upper half space is

$$\mathfrak{S}_g = \left\{ \tau \in M_g(\mathbb{C}) \;\middle|\; \tau = \tau^t \text{ and } \mathrm{Im}(\tau) \text{ is positive definite} \right\}$$

Theorem $\mathbb{C}^g / \Lambda_{\binom{\tau}{1}}$ is an abelian variety

$\iff$ $\mathbb{C}^g / \Lambda_{\binom{\tau}{1}}$ embeds in projective space

$\iff$ $\mathbb{C}^g / \Lambda_{\binom{\tau}{1}}$ has an ample line bundle

$\iff$ $\tau \in \mathfrak{S}_g$

The symplectic group $Sp_{2g}(\mathbb{R})$ is

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2g}(\mathbb{R}) \;\middle|\; \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \delta^t & -\beta^t \\ -\gamma^t & \alpha^t \end{pmatrix} = 1 \right\}$$

Then $Sp_{2g}(\mathbb{R})$ acts on $\mathfrak{S}_g$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tau = (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1}.$$

Then

$$\mathrm{Stab}_{Sp_{2g}(\mathbb{R})} \begin{pmatrix} i & & 0 \\ & \ddots & \\ 0 & & i \end{pmatrix} = Sp_{2g}(\mathbb{R}) \cap O_{2g}(\mathbb{R})$$

$$= \left\{ \begin{pmatrix} \delta & -\gamma \\ \gamma & \delta \end{pmatrix} \in M_{2g}(\mathbb{R}) \;\middle|\; \begin{matrix} \gamma\gamma^t = \delta\delta^t \text{ and} \\ \gamma\delta^t = \delta\gamma^t \end{matrix} \right\}$$

<u>Theorem</u> Let $K = Sp_{2g}(\mathbb{R}) \cap O_{2g}(\mathbb{R})$.

$$\begin{aligned} Sp_{2g}(\mathbb{R})/K &\xrightarrow{\sim} \mathcal{G}_g \\ gK &\longmapsto g\begin{pmatrix} i & & \\ & \ddots & \\ & & i \end{pmatrix} \end{aligned}$$

<u>HW</u>: Let

$$U_g(\mathbb{C}) = \{ g \in GL_g(\mathbb{C}) \mid g\bar{g}^t = 1 \}$$

Show that

$$\begin{aligned} Sp_{2g}(\mathbb{R}) \cap O_{2g}(\mathbb{R}) &\longrightarrow U_g(\mathbb{C}) \\ \begin{pmatrix} \delta & -\gamma \\ \gamma & \delta \end{pmatrix} &\longmapsto \gamma i + \delta \end{aligned}$$

is an isomorphism of $\mathbb{R}$-Lie groups.

Line bundles on a complex torus

Let $g \in \mathbb{Z}_{>0}$ and $\omega_1, \ldots, \omega_{2g} \in \mathbb{C}^g$. Let

$$\Lambda_\Omega = \mathbb{Z}\text{-span}\{\omega_1, \ldots, \omega_{2g}\}.$$

Define Pic_Ω to be the group of pairs (H, χ)

$H: \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ is a Hermitian form

$\chi: \Lambda_\Omega \to U_1(\mathbb{C})$

such that if $\gamma_1, \gamma_2 \in \Lambda_\Omega$ then

$\mathrm{Im}(H(\gamma_1, \gamma_2)) \in \mathbb{Z}$ and

$$\chi(\gamma_1 + \gamma_2) = \chi(\gamma_1)\chi(\gamma_2)e^{i\pi \mathrm{Im}(H(\gamma_1, \gamma_2))}$$

with operation given by

$$(H_1, \chi_1) + (H_2, \chi_2) = (H_1 + H_2, \chi_1 \chi_2).$$

Let $\mathbb{C}^g \xrightarrow{\pi} \mathbb{C}^g/\Lambda_\Omega$ be the quotient map.
Let $(H, \chi) \in \mathrm{Pic}_\Omega$. Define a line bundle

$\mathcal{L}_{H,\chi}$ on $\mathbb{C}^g/\Lambda_\Omega$ by

$$\mathcal{L}_{H,\chi}(U) = \left\{ f: \pi^{-1}(U) \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic} \\ f \text{ satisfies } (*) \end{array} \right\}$$

where $(*)$ is:
if $z \in \mathbb{C}^g$ and $\gamma \in \Lambda_\Omega$ then

$$f(z+\gamma) = f(z)\chi(\gamma)e^{\pi(H(z,\gamma) + \frac{1}{2}H(\gamma,\gamma))}$$

<u>Theorem</u> (Appell-Humbert Theorem)

(see [Shimizu-Ueno Theorem 2.4], [Igusa Chap. II] and [Harder §5.2.1])

As $\mathbb{Z}$-modules

$$\mathrm{Pic}_\Omega \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{line bundles on } \mathbb{C}^g/\Lambda_\Omega \end{matrix} \right\}$$

$$(H, \chi) \longmapsto \mathcal{L}_{H,\chi}$$

Let $(H, \chi) \in \mathrm{Pic}_\Omega$ and define

$$\langle\ ,\ \rangle : \Lambda_\Omega \times \Lambda_\Omega \to \mathbb{Z} \quad \text{by} \quad \langle \gamma_1, \gamma_2 \rangle = \mathrm{Im}(H(\gamma_1, \gamma_2))$$

and let

$E \in M_{2g\times 2g}(\mathbb{Z})$ be given by $E_{ij} := \langle \omega_i, \omega_j \rangle$.

<u>Theorem</u> $\mathcal{L}_{H,\chi}$ is ample

$\Leftrightarrow$ $H: \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ is positive definite

$\Leftrightarrow$ E satisfies

$$E = -E^t,$$

$$\Omega^t E^{-1} \Omega = 0, \text{ and}$$

$$i\,\Omega^t E^{-1} \overline{\Omega} \in \mathbb{R}_{>0}.$$

Change basis of Λ_Ω so that

$$E = \left(\begin{array}{c|c} 0 & \begin{matrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_g \end{matrix} \\ \hline \begin{matrix} -d_1 & & 0 \\ & -d_2 & \\ 0 & & \ddots \; -d_g \end{matrix} & 0 \end{array}\right) \quad \text{with} \quad d_i \in \mathbb{Z}_{>0}$$

$$d_g\mathbb{Z} \subseteq \cdots \subseteq d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$$

This means ~~we~~ let $M \in GL_{2g}(\mathbb{Z})$ and consider $\Lambda_{M\Omega} = \Lambda_\Omega$. Write $M\Omega = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$

~~and~~ The conditions on E give $\det W_2 \neq 0$ and

$$M\Omega W_2^{-1} = \begin{pmatrix} \tau\Delta^{-1} \\ 1 \end{pmatrix} \quad \text{with} \quad \tau \in G_g \text{ and } \Delta = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_g \end{pmatrix}$$

Let

$$Sp(\Delta, \mathbb{Z}) = \{ M \in GL_{2g}(\mathbb{Z}) \mid MEM^t = E \}$$

<u>Theorem</u>

$$Sp(\Delta,\mathbb{Z}) \backslash G_g \xleftrightarrow{\sim} \left\{ \begin{matrix} \Delta\text{-polarized} \\ \text{abelian varieties} \end{matrix} \right\}$$

$$\begin{pmatrix} \tau\Delta^{-1} \\ 1 \end{pmatrix} \longmapsto \left(\mathbb{C}^g / \Lambda_{M\Omega W_2^{-1}}, \mathcal{L}_{(H,\alpha)} \right)$$

~~where~~ An <u>polarized abelian variety</u> $(\mathbb{C}^g/\Lambda, \mathcal{L})$ is <u>principally polarized</u> if $\Delta = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Embedding in projective space [Harder §5.2.4]

Theorem 5.2.19 (Kodaira embedding Theorem)

X is a compact complex manifold.

T_X is the tangent bundle.

Let $h: T_X \times T_X \to \mathbb{C}$ be a Hermitian metric.

Assume that the corresponding 2-form ω_h satisfies

ω_h is integral, i.e. $\omega_h \in H^2(X, \mathbb{Z})$

Then there exists an ample line bunde

$\mathcal{L}$ on X (with $c_1(\mathcal{L}) = \omega_h$)

and there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq N}$ then

$$\begin{matrix} X & \longrightarrow & \mathbb{P}(H^0(X, \mathcal{L}^{\otimes n})) \\ p & \longmapsto & H_p \end{matrix}$$ is an embedding

where

$$H_p = \{ s \in H^0(X, \mathcal{L}^{\otimes n}) \mid s(p) = 0 \}$$

(a codimension 1 $\mathbb{C}$-submodule of $H^0(X, \mathcal{L}^{\otimes n})$).

<u>The homogeneous coordinate ring</u>

(see [Harder §5.1.7 and §5.1.8] and [Igusa Ch. III §6])

Let $\mathbb{C}^g/\Lambda$ be an abelian variety with ample line bundle $\mathcal{L}$. The <u>homogeneous</u> <u>coordinate ring</u> of $(\mathbb{C}^g/\Lambda, \mathcal{L})$ is

$$\Theta[\mathbb{C}^g/\Lambda, \mathcal{L}] = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(\mathbb{C}^g/\Lambda, \mathcal{L}^{\otimes d})$$

A <u>Riemann-Theta function</u> is an element of $\Theta[\mathbb{C}^g/\Lambda, \mathcal{L}]$. The Jacobi theta functions are the case $g=1$.

Let

$\{x_1, \ldots x_k\}$ be a basis of $H^0(\mathbb{C}^g/\Lambda, \mathcal{L})$

$\{y_1, \ldots y_l\}$ a basis of $H^0(\mathbb{C}^g/\Lambda, \mathcal{L}^{\otimes 2})$

$\{z_1, \ldots, z_m\}$ a basis of $H^0(\mathbb{C}^g/\Lambda, \mathcal{L}^{\otimes 3})$

Then

$$\mathbb{C}[x_1, \ldots x_k, y_1, \ldots, y_l, z_1, \ldots, z_m] \twoheadrightarrow \Theta[\mathbb{C}^g/\Lambda, \mathcal{L}].$$