Algebraic Geometry Week 9 Friday

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Theta functions

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Abelian varieties $\mathbb{C}^g/\Lambda_\tau$ are indexed by $\tau \in \mathcal{G}_g$.

Functions (and sections of line bundles) on $\mathbb{C}^g/\Lambda_\tau$ are functions on $\mathbb{C}^g$ with some Λ_τ-periodicity.

[Ha, Prop. 5.2.33] A Riemann theta function for $(\mathbb{C}^g/\Lambda_\tau, \mathcal{L})$ is an element of $H^0(\mathbb{C}^g/\Lambda_\tau, \mathcal{L}^{\otimes d})$

(Note: Harder takes the principal polarization for which $\dim(H^0(\mathbb{C}^g/\Lambda_\tau, \mathcal{L}^{\otimes d})) = d^g$)

The <u>Riemann theta function</u> $\theta : \mathbb{C}^g \times \mathcal{G}_g \to \mathbb{C}$ is

$$\theta(z,\tau) = \sum_{\ell \in \mathbb{Z}^g} e^{2\pi i (\frac{1}{2}\ell\tau\ell^t + \ell z^t)}$$

(see [Shimizu-Ueno (2.3.4)]. Then

$$\theta(z,\tau) \in H^0(\mathbb{C}^g/\Lambda_\tau, \mathcal{L}_{H,1})$$

The <u>Jacobi theta function</u> $\theta : \mathbb{C} \times \mathcal{G}_1 \to \mathbb{C}$ is

$$\theta(z,q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} e^{2\pi i n z}, \quad \text{where } q = e^{i\pi\tau}$$

(see [AAR (10.7.4)] and [TF, App I (I.1)] and this is $\theta_{1,1}(u,p)$ in [Harder (5.147)])

Elliptic Functions

Let $\Lambda = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\}$ be a rank 2 lattice in $\mathbb{C}$.
We prefer $\Lambda_\tau = \mathbb{Z}\text{-span}\{1, \tau\}$ with $\tau \in G_1$.

An elliptic function relative to Λ is a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ such that if $\gamma \in \Lambda$ and $z \in \mathbb{C}$ then $f(z+\gamma) = f(z)$.

The Weierstrass $\wp$-function is

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

HW! Show that

$$\wp'(z;\tau) = \frac{d}{dz}\wp(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$$

Theorem The ring of elliptic functions is

$$\mathbb{C}(\wp(z;\tau), \wp'(z;\tau))$$

Theorem

$$\begin{aligned} \iota_\tau : \mathbb{C}/\Lambda_\tau &\xrightarrow{\sim} \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto [\wp(z), \wp'(z), 1] \end{aligned}$$

is an embedding of $\mathbb{C}/\Lambda_\tau$ into projective space.

The expansion of $\wp$

Let $k \in \mathbb{Z}_{\geq 2}$. The Eisenstein series of weight $2k$ is

$$G_{2k}(\tau) = \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \frac{1}{\omega^{2k}}$$

Let

$$g_2(\tau) = 60\, G_4(\tau) \quad \text{and} \quad g_3(\tau) = 140\, G_6(\tau).$$

Theorem

(a) $\wp(z,\tau) = z^{-2} + \sum_{k \in \mathbb{Z}_{>0}} (2k+1)\, G_{2k+2}(\tau)\, z^{2k}$

(b) $(\wp'(z,\tau))^2 = 4\,\wp(z,\tau)^3 - g_2(\tau)\wp(z,\tau) - g_3(\tau)$

(c) $\mathrm{im}(\iota_\tau) = \{[0,1,0]\} \cup \{[x,y,1] \in \mathbb{P}^2 \mid y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)\}$

(d) The discriminant of $4x^3 - g_2(\tau)x - g_3(\tau)$ is

$$\Delta(\tau) = g_2(\tau)^3 - 27 g_3(\tau)^2 = \begin{vmatrix} 1 & 1 & 1 \\ \wp(1,\tau) & \wp(\tau,\tau) & \wp(-1-\tau,\tau) \\ \wp(1,\tau)^2 & \wp(\tau,\tau)^2 & \wp(-1-\tau,\tau)^2 \end{vmatrix}$$

(e) $4x^3 - g_2(\tau)x - g_3(\tau) = 4(x - \wp(1,\tau))(x - \wp(\tau,\tau))(x - \wp(-1-\tau,\tau))$

(see [Silverman Ch. I Proof of Theorem 8.1] and [Whittaker-Watson 20.32])

The j-function is $j(\tau) = 1728 \dfrac{g_2(\tau)^3}{\Delta(\tau)}$.

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Jacobi theta functions (see [Andrews-Askey-Roy (10.7.4) and (10.7.7)])

Let $\theta : \mathbb{C}\times G_1 \to \mathbb{C}$ be given by

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} e^{2\pi i n z}, \text{ where } q = e^{i\pi\tau}$$

Let

$$\theta_4(z,\tau) = \theta(z,\tau)$$

$$\theta_1(z,\tau) = \frac{1}{i} q^{1/4} e^{i\pi z}\theta(z+\tfrac{\tau}{2},\tau)$$

$$\theta_2(z,\tau) = q^{1/4}e^{i\pi z}\theta(z+\tfrac{1}{2}+\tfrac{\tau}{2},\tau)$$

$$\theta_3(z,\tau) = \theta(z+\tfrac{1}{2},\tau).$$

In Harder's notation (CHECK THIS)

$$\theta_4 = \theta_{0,1}, \quad \theta_1 = \theta_{0,0}, \quad \theta_2 = \theta_{1,0}, \quad \theta_3 = \theta_{1,1}$$

Addition formulas for theta functions

Fix $\tau\in G_1$.

$$\theta_1(u)\theta_1(v)\theta_1(w)\theta_1(u+v+w) + \theta_4(u)\theta_4(v)\theta_4(w)\theta_4(u+v+w)$$
$$= \theta_4(0)\theta_4(u+v)\theta_4(u+w)\theta_4(v+w)$$

$$\theta_1(u)\theta_1(v)\theta_1(w)\theta_4(u+v+w) + \theta_4(u)\theta_4(v)\theta_4(w)\theta_1(u+v+w)$$
$$= \theta_4(0)\theta_4(u+v)\theta_1(u+w)\theta_1(v+w)$$

$$\theta_4(u-v)\theta_1(u+v) - \theta_4(u+v)\theta_1(u-v)$$
$$= \frac{2\theta_1(v)\theta_2(u)\theta_3(u)\theta_4(v)}{\theta_2(0)\theta_3(0)}.$$

Relation between $\wp$ and θ

$$\wp(z;\tau) = (\text{const})\cdot \frac{\theta_2(z,\tau)^2}{\theta_1(z,\tau)^2}\frac{\theta_3(0,\tau)^2}{\theta_4(0,\tau)^2} + (\text{constant})$$

(see [Whittaker-Watson §21.7.3])

The functions sn, cn, dn

Let $k = \dfrac{\theta_2(0,\tau)^2}{\theta_3(0,\tau)^2}$ and $k' = \dfrac{\theta_4(0,\tau)^2}{\theta_3(0,\tau)^2}$

Let $v = \dfrac{1}{\pi\theta_3(0)^2} z$ and define

$$sn(z,\tau) = \sqrt{\frac{1}{k}}\,\frac{\theta_1(v,\tau)}{\theta_4(v,\tau)}, \qquad cn(z,\tau) = \sqrt{\frac{k'}{k}}\,\frac{\theta_2(v,\tau)}{\theta_4(v,\tau)}$$

$$dn(z,\tau) = \sqrt{k'}\,\frac{\theta_3(v,\tau)}{\theta_4(v,\tau)}.$$

Theorem

(a) $cn^2(z,\tau) + sn^2(z,\tau) = 1$

(b) $dn^2(z,\tau) + k^2 sn^2(z,\tau) = 1$

(c) $\dfrac{d}{dz} sn(z) = cn(z)\, dn(z)$

(d) $y = sn(z)$ satisfies $\left(\dfrac{dy}{dz}\right)^2 = (1-y^2)(1-k^2y^2)$

(e) $y = sn(z)$ satisfies $y'' + (1+k^2)y - 2k^2y^3 = 0$.