# Algebraic Geometry Week 6

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## Vector bundles: the equivalences

$$\left\{ \text{vector bundles } \begin{matrix} M \\ \pi\downarrow \\ X \end{matrix} \text{ of dimension } n \right\} \qquad M$$

$$\updownarrow$$

$$\{ \text{locally free } \mathcal{O}_X\text{-modules } \mathcal{M} \} \qquad \mathcal{M}$$

$$\updownarrow$$

$$\{ 1\text{ cocycles on } X \text{ valued in } Gl_n(\mathbb{C}) \} \quad g = (g_{uv})$$

giving

$$\left\{ \begin{matrix} \text{isomorphism classes} \\ \text{of vector bundles of} \\ \text{dimension } n \text{ on } X \end{matrix} \right\} \longleftrightarrow H^1(X, C^0(X, Gl_n(\mathbb{C}))).$$

where

$$\mathcal{M}(U) = \left\{ s : U \to M \;\middle|\; \begin{matrix} s \text{ is continuous and} \\ \pi \circ s = id_U \end{matrix} \right\}$$

and, if $\mathcal{S}$ is an open cover of $X$ then

$$\varphi_u \circ \varphi_v^{-1} : (U \cap V) \times \mathbb{C}^n \longrightarrow (U \cap V) \times \mathbb{C}^n$$
$$(p, v) \longmapsto (p, g_{uv}(p) v)$$

Essentially

$$M = Spec(\mathcal{M}).$$

if one can make proper sense of this statement.

## Vector bundle: Definition 0

A vector bundle of rank $n$ on $X$ is an extension

$$0 \to \mathbb{C}^n \to M \to X \to 0$$

The trivial bundle is $M \cong X \times \mathbb{C}^n$

$$0 \to \mathbb{C}^n \to X \times \mathbb{C}^n \to X \to 0.$$

## Vector bundle: Definition 1

(Harder §6.2.3 and §4.3.1 and §4.3.2)

Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space.
A vector bundle of rank $n$ on $X$ is a locally free sheaf $\mathcal{M}$ of rank $n$ on $X$.

A line bundle on $X$ is a vector bundle of rank 1 on $X$.

A locally free sheaf on $X$ is a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-module such that

if $p \in X$ then there exists $U \in \mathcal{T}_X$ with $p \in U$ and a sheaf isomorphism

$$\mathcal{M}_U \cong \mathcal{O}_U^{\oplus n}$$

The global sections of $\mathcal{M}$ is

$$H^0(X, \mathcal{M}) = \mathcal{M}(X).$$

## Vector bundle: Definition 2

Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space. A <u>vector bundle of rank $n$ on $X$</u> is a topological space $(M, \mathcal{T}_X)$ with a continuous map

$$\begin{matrix} (M, \mathcal{T}_M) \\ \pi \downarrow \\ (X, \mathcal{T}_X) \end{matrix}$$

such that there exists an open cover $\mathcal{S} \subseteq \mathcal{T}_X$ and homeomorphisms

$$\begin{array}{ccc} V \times \mathbb{C}^n & \xrightarrow{\varphi_V} & \pi^{-1}(V) \\ (p, a_1, \ldots, a_n) & \longmapsto & a_1 e_1^V(p) + \cdots + a_n e_n^V(p) \end{array} \qquad \text{for } V \in \mathcal{S}$$

such that if $U, V \in \mathcal{S}$ and $p \in U \cap V$ then

$$g_{UV}(p) \in GL_n(\mathbb{C}),$$

where

$$\begin{array}{rccc} \varphi_U^{-1} \circ \varphi_V : & (U \cap V) \times \mathbb{C}^n & \longrightarrow & (U \cap V) \times \mathbb{C}^n \\ & (p, a) & \longmapsto & (p, g_{UV}(p)\, a) \end{array}$$

The $\varphi_V$ are <u>local trivializations</u>.

A <u>global section</u> of $\begin{matrix} M \\ \downarrow \pi \\ X \end{matrix}$ is a continuous map $\begin{matrix} M \\ \uparrow s \\ X \end{matrix}$ such that $\pi \circ s = id_X$.

$$H^0(G/B, M) = \left\{ s : X \to M \;\middle|\; \begin{array}{l} s \text{ is continuous and} \\ \pi \circ s = id_M \end{array} \right\}$$

Let $M \xrightarrow{\pi} X$ be a vector bundle on $X$.

The <u>sheaf of sections</u> of $M$ is given by

$$\mathcal{M}(U) = \left\{ s : U \to M \;\middle|\; \begin{array}{l} s \text{ is continuous and} \\ \pi \circ s = id_U \end{array} \right\}$$

with $\mathcal{O}_X(U)$-module structure given by

$$(s_1 + s_2)(p) = s_1(p) + s_2(p) \text{ and } (fs)(p) = f(p)s(p).$$

The existence of the local trivializations $\varphi_V$

gives $\mathcal{M}(U) \simeq \mathcal{O}_X(U)^{\oplus n}$ for $U \in \mathcal{S}$.

So $\mathcal{M}$ is a locally free sheaf on $X$.

Vector bundles: Definition 3

Let $C^0(X, Gl_n(\mathbb{C}))$ be the sheaf of continuous functions $X \to Gl_n(\mathbb{C})$, where $Gl_n(\mathbb{C})$ has the topology coming from $\mathbb{C}$ via $Gl_n(\mathbb{C}) \subseteq \mathbb{C}^{n^2}$

Let $\mathcal{S}$ be an open cover of $X$. An $\mathcal{S}$ 1-cocycle is a collection of continuous maps

$$g_{UV} : U \cap V \longrightarrow GL_n(\mathbb{C}) \text{ for } U, V \in \mathcal{S}$$

such that

(a) if $U \in \mathcal{S}$ and $p \in U$ then $g_{UU}(p) = 1$,

(b) If $U, V, W \in \mathcal{S}$ and $p \in U \cap V \cap W$ then

$$g_{UV}(p) g_{VW}(p) = g_{UW}(p).$$

A vector bundle of rank $n$ on $X$ is an $\mathcal{S}$ 1-cocycle $g = (g_{UV})_{U,V \in \mathcal{S}}$ for an open cover $\mathcal{S}$.

A global section of $g = (g_{UV})_{U,V \in \mathcal{S}}$ is a collection of continuous maps

$$s_U : U \to \mathbb{C}^n \quad \text{for } U \in \mathcal{S}$$

such that

(a) if $U, V \in \mathcal{S}$ and $p \in U \cap V$ then

$$g_{UV}(p) s_V(p) = s_U(p).$$

An $\mathcal{S}$ change of trivialization on $X$ is a collection of continuous maps

$$h_V : V \longrightarrow GL_n(\mathbb{C}) \quad \text{for } V \in \mathcal{S}$$

Let

$Z^1(X, \mathcal{S}, C^0(X, GL_n(\mathbb{C}))) = \{\mathcal{S}$ 1-cocycles on $X\}$

$B^1(X, \mathcal{S}, C^0(X, GL_n(\mathbb{C}))) = \{\mathcal{S}$ change of trivializations on $X\}$

and

$$H^1(X, \mathcal{S}, C^0(X, GL_n(\mathbb{C}))) = \frac{Z^1(X, \mathcal{S}, C^0(X, GL_n(\mathbb{C})))}{B^1(X, \mathcal{S}, C^0(X, GL_n(\mathbb{C})))}$$

$$= \frac{\{g = (g_{UV}) \mid g \in Z^1(X, \mathcal{S}, C^0(X, GL_n(\mathbb{C})))\}}{\left\langle g = g' \;\middle|\; \begin{array}{l}\text{there exists } h \in B^1(X, \mathcal{S}, C^0(X, GL_n(\mathbb{C}))) \\ \text{such that } g'_{UV} = h_U g_{UV} h_V^{-1} \text{ for } U, V \in \mathcal{S}\end{array} \right\rangle}$$

Let

$$H^1(X, C^0(X, GL_n(\mathbb{C}))) = \varinjlim_{\mathcal{S}} H^1(X, \mathcal{S}, C^0(X, GL_n(\mathbb{C})))$$

Then

$$\left\{\begin{array}{c}\text{isomorphism classes} \\ \text{of vector bundles} \\ \text{of rank } n \text{ on } X\end{array}\right\} = H^1(X, C^0(X, GL_n(\mathbb{C})))$$

Principal G-bundles (see Harder Definition 3.11 and §6.2.4)

$$\{\text{principal } G\text{-bundles on } X\} \qquad \begin{array}{c} P \\ \pi\downarrow \\ X \end{array}$$
$$\updownarrow$$
$$\{1\text{ cocycles on } X \text{ valued in } G\} \qquad g = (g_{UV})$$

where

$$\varphi_U \circ \varphi_V^{-1} : (U \cap V) \times GL_n(\mathbb{C}) \longrightarrow (U \cap V) \times GL_n(\mathbb{C})$$
$$(p, h) \longmapsto (p, g_{UV}(p) h)$$

gives
$$\left\{\begin{array}{c}\text{isomorphism classes} \\ \text{principal } G\text{-bundles} \\ \text{on } X\end{array}\right\} \longleftrightarrow H^1(X, G).$$

And, identifying

$$GL_n(\mathbb{C}) = \{\text{bases on } \mathbb{C}^n\} = B(\mathbb{C}^n)$$

gives an equivalence

$$\left\{\begin{array}{c}\text{isomorphism classes} \\ \text{vector bundles of} \\ \text{rank } n \text{ on } X\end{array}\right\} \qquad U \times \mathbb{C}^n$$
$$\updownarrow \qquad\qquad \downarrow$$
$$\left\{\begin{array}{c}\text{isomorphism classes} \\ \text{of principal } GL_n\text{-bundles} \\ \text{on } X\end{array}\right\} \qquad U \times GL_n(\mathbb{C})$$

via setting

$$B(U \times \mathbb{C}^n) = \left\{ (e_1, \ldots, e_n) \,\middle|\, \begin{array}{l}\text{sections of } M(U) \text{ which} \\ \text{are } \mathcal{O}_X(U)\text{-linearly indep.}\end{array}\right\}$$
$$= \{\text{trivializations over } U\} = \{\text{bases in } \mathbb{C}^n \text{ over } U\}.$$

## Classifying vector bundles on $\mathbb{P}^1$

The scheme $\mathbb{P}^1$ is defined by two open sets

$$U_0 = \{[c,1] \mid c \in \mathbb{C}\} \simeq \mathbb{C} \quad \text{with} \quad \mathcal{O}_{U_0} = \mathbb{C}[z]$$

$$U_\infty = \{[1,c] \mid c \in \mathbb{C}\} \simeq \mathbb{C} \quad \text{with} \quad \mathcal{O}_{U_1} = \mathbb{C}[z^{-1}]$$

with

$$U_0 \cap U_\infty = \{[c,1] \mid c \in \mathbb{C}^\times\} = \{[1,c^{-1}] \mid c \in \mathbb{C}^\times\} \simeq \mathbb{C}[z,z^{-1}]$$

and

$$\mathrm{res}^{U_0}_{U_0\cap U_\infty} : \begin{matrix} \mathbb{C}[z] & \longrightarrow & \mathbb{C}[z,z^{-1}] \\ f & \longmapsto & f \end{matrix} \qquad \mathrm{res}^{U_\infty}_{U_0\cap U_\infty} : \begin{matrix} \mathbb{C}[z^{-1}] & \longrightarrow & \mathbb{C}[z,z^{-1}] \\ g & \longmapsto & g \end{matrix}$$

A vector bundle of rank $n$ on $\mathbb{P}^1$ is defined by

$$g_{0\infty} : U_0 \cap U_\infty \longrightarrow GL_n(\mathbb{C})$$

where

$$\begin{matrix} \varphi_\infty \circ \varphi_0^{-1} : & (U_0 \cap U_\infty) \times \mathbb{C}^n & \longrightarrow & (U_\infty \cap U_0) \times \mathbb{C}^n \\ & (z, v) & \longmapsto & (z^{-1}, g_{0\infty}(z) v) \end{matrix}$$

So a vector bundle of rank $n$ on $\mathbb{P}^1$ is defined by a matrix $g_{0\infty} \in GL_n(\mathbb{C}[z,z^{-1}])$.

Here $\begin{matrix} M \\ \pi \downarrow \\ \mathbb{P}^1 \end{matrix}$ is the vector bundle and

$$\varphi_0 : U_0 \times \mathbb{C}^n \to \pi^{-1}(U_0) \quad \text{and} \quad \varphi_\infty : U_\infty \times \mathbb{C}^n \to \pi^{-1}(U_\infty)$$

are the trivializations over $U_0$ and $U_\infty$.

A change of trivialization for $S = (\mathcal{U}_0, \mathcal{U}_\infty)$ on $\mathbb{P}^1$ is $h = (h_0, h_\infty)$ with

$$h_0 : \mathcal{U}_0 \longrightarrow GL_n(\mathbb{C}) \qquad \text{and} \qquad h_\infty : \mathcal{U}_\infty \longrightarrow GL_n(\mathbb{C})$$
$$z \longmapsto h_0(z) \qquad\qquad z^{-1} \longmapsto h_\infty(z^{-1})$$

So a change of trivialization is a pair

$$h = (h_0, h_\infty) \quad \text{with} \quad h_0 \in GL_n(\mathbb{C}[z]), \quad h_\infty \in GL_n(\mathbb{C}[z^{-1}]).$$

Thus

$$\left\{\begin{matrix}\text{isomorphism classes} \\ \text{of vector bundles of} \\ \text{rank } n \text{ on } \mathbb{P}^1\end{matrix}\right\} \leftrightarrow GL_n(\mathbb{C}[z]) \backslash GL_n(\mathbb{C}[z, z^{-1}]) / GL_n(\mathbb{C}[z^{-1}])$$

Note that if $g \in GL_n(\mathbb{C}[z, z^{-1}])$ then

$$\det g \in \mathbb{C}[z, z^{-1}]^\times = \{ c z^d \mid c \in \mathbb{C}^\times \text{ and } d \in \mathbb{Z} \}$$

If $h_0 \in GL_n(\mathbb{C}[z])$ and $h_\infty \in GL_n(\mathbb{C}[z^{-1}])$ then

$$\det h_0 \in \mathbb{C}[z]^\times = \mathbb{C}^\times \quad \text{and} \quad \det h_\infty \in \mathbb{C}[z^{-1}]^\times = \mathbb{C}^\times.$$

<u>Proposition</u> Let $G = GL_n(\mathbb{C}[z, z^{-1}])$,

$K^+ = GL_n(\mathbb{C}[z])$ and $K^- = GL_n(\mathbb{C}[z^{-1}])$

Let
$$t_\lambda = \begin{pmatrix} z^{\lambda_1} & & \\ & \ddots & \\ & & z^{\lambda_n} \end{pmatrix} \quad \text{for } \lambda_1, \ldots, \lambda_n \in \mathbb{Z}.$$

Then
$$GL_n(\mathbb{C}[z, z^{-1}]) = \bigsqcup_{\substack{\lambda_1, \ldots, \lambda_n \in \mathbb{Z} \\ \lambda_1 \geq \cdots \geq \lambda_n}} K^+ t_\lambda K^-$$

<u>Corollary</u> If $M$ is a vector bundle of rank $n$ on $\mathbb{P}^n$ then there exist unique $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that

$$M \simeq \mathcal{O}(\lambda_1) \oplus \cdots \oplus \mathcal{O}(\lambda_n),$$ (Harder Theorem 9.4.2)

where
$$\begin{aligned} \mathrm{Pic}(\mathbb{P}^1) &\simeq \mathbb{Z} \\ \mathcal{O}(d) &\longleftarrow d \end{aligned}$$

# Building vector bundles on $G/B$

Let $G = GL_n(\mathbb{C})$ and $B = \left\{ \begin{matrix} \text{upper triangular} \\ \text{matrices in } GL_n(\mathbb{C}) \end{matrix} \right\}$

Let $V$ be a $B$-module. Define

$$G \times_B V = \frac{G \times V}{\langle (gb, v) = (g, bv) \mid g \in G, b \in B, v \in V \rangle}$$

with the quotient topology for $G \times V \to G \times_B V$.

Define

$$\begin{array}{ccc} G \times_B V & & [g, v] \\ \pi \downarrow & & \downarrow \\ G/B & & gB \end{array}$$

Then $\begin{array}{c} G \times_B V \\ \pi \downarrow \\ G/B \end{array}$ is a vector bundle on $G/B$ of rank $\dim(V)$. Let $\mathcal{M}$ be the sheaf on $G/B$ given by

$$\mathcal{M}(\mathcal{U}) = \left\{ \begin{matrix} s : \mathcal{U} \to G \times_B V \\ gB \mapsto [g, s(gB)] \end{matrix} \;\middle|\; \begin{matrix} s \text{ is continuous and} \\ \pi \circ s = id_{\mathcal{U}} \end{matrix} \right\}$$

with $\mathcal{O}_{G/B}(\mathcal{U})$-action given by

$$(fs)(gB) = [g, f(gB)s(gB)]$$

Note that

$$(\pi \circ fs)(gB) = \pi(fs(gB)) = \pi([g, f(gB)s(gB)]) = gB.$$

## Line bundles on $G/B$

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$. Define a group homomorphism

$$X^\lambda : B \to \mathbb{C}^\times \quad \text{by} \quad X^\lambda\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = a_1^{\lambda_1} \cdots a_n^{\lambda_n}$$

and a 1-dimensional $B$-module

$$\mathbb{C}_\lambda = \mathbb{C}\text{span}\{v_\lambda\} \quad \text{with} \quad bv_\lambda = X^\lambda(b)v_\lambda$$

for $b \in B$. Let $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$, where

$$\begin{array}{ccc} G \times_B \mathbb{C}_\lambda & & [g, cv_\lambda] \\ \pi \downarrow & & \downarrow \\ G/B & & gB \end{array} \qquad \text{for } g \in G,\ c \in \mathbb{C}.$$

A global section of $\mathcal{L}_\lambda$ is

$$\begin{aligned} s : G/B &\longrightarrow G \times_B \mathbb{C}_\lambda \\ gB &\longmapsto [g, s(g)v_\lambda] \end{aligned}$$

so that $s$ is identified with a function

$$\begin{aligned} s : G &\longrightarrow \mathbb{C} \\ g &\longmapsto s(g) \end{aligned} \qquad \text{such that } s(gb) = s(g)X^\lambda(b^{-1}).$$

where the condition comes from

$$[g, s(g)v_\lambda] = [gb, s(gb)v_\lambda] = [g, s(gb)bv_\lambda] = [g, s(gb)X^\lambda(b)v_\lambda],$$

# The line bundles $\mathcal{L}_\lambda$ on $\mathbb{P}^1 = GL_2(\mathbb{C})/B$

Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ and

$$X^\lambda : B \to \mathbb{C}^\times \quad \text{given by} \quad X^\lambda \begin{pmatrix} a_1 & c \\ 0 & a_2 \end{pmatrix} = a_1^{\lambda_1} a_2^{\lambda_2}.$$

Identify

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} B \in GL_2(\mathbb{C})/B \quad \text{with} \quad [a, c] \in \mathbb{P}^1.$$

Then

$$U_0 = \left\{ \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} B \mid c \in \mathbb{C} \right\} = \{ [c, 1] \mid c \in \mathbb{C} \} \cong \mathbb{C}$$

$$U_\infty = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} B \mid c \in \mathbb{C} \right\} = \{ [1, c] \mid c \in \mathbb{C} \} \cong \mathbb{C}$$

and, if $c \neq 0$ then

$$[c, 1] = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} \begin{pmatrix} c & -1 \\ 0 & c^{-1} \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} B = [1, c^{-1}]$$

and a global section $s : \mathbb{P}^1 \to \mathcal{L}_\lambda$ is identified with a function $s : G \to \mathbb{C}$ satisfying

$$s(gb) = s(g) X^\lambda(b^{-1}) \quad \text{for } g \in G,\ b \in B$$

so that, if $c \neq 0$ then

$$s([c,1]) = s\begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} = s\left( \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} \begin{pmatrix} c & -1 \\ 0 & c^{-1} \end{pmatrix} \right)$$

$$= s\begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} X^\lambda \begin{pmatrix} c^{-1} & +1 \\ 0 & c \end{pmatrix} = c^{\lambda_2 - \lambda_1} s([1, c^{-1}])$$

So $\mathcal{L}_\lambda \cong \mathcal{O}(\lambda_2 - \lambda_1)$

A. Ram

# The Borel-Weil-Bott theorem

Let $G = GL_n(\mathbb{C})$, $B = \left\{ \begin{matrix} \text{upper triangular} \\ \text{matrices in } G \end{matrix} \right\}$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$. Define $X^\lambda : B \to \mathbb{C}^\times$ by

$$X^\lambda \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}.$$

and let

$$\begin{array}{c} \mathcal{L}_\lambda = G \times_B \mathbb{C} v_\lambda = \dfrac{G \times \mathbb{C} v_\lambda}{\langle [gb, cv_\lambda] = [g, c X^\lambda(b) v_\lambda] \text{ for } g \in G, c \in \mathbb{C}, b \in B \rangle} \\ \pi \big\downarrow \\ G/B \end{array}$$

where $\pi([g, cv_\lambda]) = gB$. Identify global sections of $\mathcal{L}_\lambda$ with functions

$$s : G \to \mathbb{C} \quad \text{such that } s(gb) = s(g) X^\lambda(b^{-1})$$

and let

$$H^0(G/B, \mathcal{L}_\lambda) = \{\text{global sections of } \mathcal{L}_\lambda\}.$$

The group $G$ acts on $H^0(G/B, \mathcal{L}_\lambda)$ by

$$(gs)(h) = s(g^{-1}h), \quad \text{for } g \in G, h \in G.$$

So $H^0(G/B, \mathcal{L}_\lambda)$ is an $G$-module.