Sketch of proof that $G = \bigsqcup_{w\in S_n} BwB$ and

$$BwB = \{ y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)B \mid c_1,\dots,c_\ell \in \mathbb{C}\}$$

if $w = s_{i_1}\cdots s_{i_\ell}$ is a reduced word.

Recall that

$$y_i(c) = \begin{array}{c} \\ \\ i \\ i+1 \\ \\ \end{array}\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \quad (\text{columns } i,\ i+1)$$

and $B = \left\{ \begin{array}{l} \text{upper triangular} \\ \text{matrices} \\ \text{in } GL_n(\mathbb{C}) \end{array} \right\}$

Then

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right)\begin{pmatrix} 7 & 6 & 2 & 4 \\ 8 & 6 & 3 & 5 \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right)\begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{6}{8} & \frac{-5}{8} & \frac{-3}{8} \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

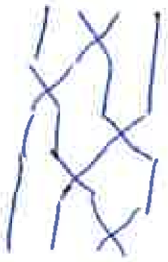
$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right)\begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{6}{8} & \frac{-5}{8} & \frac{-3}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{-5}{8} & \frac{-49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) y_2\left(\tfrac{6}{8}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{11}{8} & -\frac{15}{8} \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) y_2\left(\tfrac{6}{8}\right) y_3\left(\tfrac{11}{5}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & \frac{464}{40} \end{pmatrix}$$

$$\in y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) y_2\left(\tfrac{6}{8}\right) y_3\left(\tfrac{11}{5}\right) B$$

which is in the cell BwB which has

$w = s_2 s_1 s_3 s_2 s_3 =$  $=$ 

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Covering G/B with $\mathcal{U}_w$ such that $\mathcal{U}_w \cong \mathbb{C}^{\binom{n}{2}}$

A. Ram

Let

$w_0 =$  $=$

$$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} = \begin{array}{c} \{1,\ldots,n\} \to \{1,\ldots,n\} \\ 1 \mapsto n \\ 2 \mapsto n-1 \\ \vdots \\ n-1 \mapsto 2 \\ n \mapsto 1 \end{array}$$

Then

$$w_0 = (s_{n-1}\cdots s_2 s_1)(s_{n-1}\cdots s_3 s_2)\cdots(s_{n-1}s_{n-2})s_{n-1}$$

is a reduced word for w_0 and

$$\ell(w_0) = (n-1)+(n-2)+\cdots+2+1 = \frac{n(n-1)}{2} = \binom{n}{2}$$

Let

$$B^- = w_0 B w_0 = \left\{ \begin{array}{l} \text{lower triangular} \\ \text{matrices in } GL_n(\mathbb{C}) \end{array} \right\}$$

Define

$$\mathcal{U}_w = wB^-B$$

Since $Bw_0B \cong \mathbb{C}^{\binom{n}{2}}$ then

$$\mathcal{U}_w = wB^-B = ww_0Bw_0B \cong \mathbb{C}^{\binom{n}{2}}$$

The $\mathcal{U}_w$, $w \in S_n$ form an open cover of G/B.

Algebraic Geometry Week 4 18.08.2018 (4)
Line bundles on flag varieties UniMelb
A. Ram

Theorem $\mathbb{P}^{n-1}$

(a) $\mathrm{Pic}(\mathbb{P}^{n-1}) \cong \mathbb{Z}$
$$\mathcal{L}_d \longleftrightarrow d$$

(b) If $d \in \mathbb{Z}_{\geq 0}$ then, as $GL_n(\mathbb{C})$-modules
$$H^0(\mathbb{P}^{n-1}; \mathcal{L}_d) \cong \mathrm{Sym}^d(\mathbb{C}^n)$$
If $d \notin \mathbb{Z}_{\geq 0}$ then $H^0(\mathbb{P}^{n-1}, \mathcal{L}_d) = 0$.

(c) If $d \in \mathbb{Z}_{\geq 0}$ then
$$\dim(H^0(\mathbb{P}^{n-1}, \mathcal{L}_d)) = \binom{n+d-1}{d}$$

Theorem Fl

(a) $\mathrm{Pic}(Fl) \cong \mathbb{Z}^n$

(b) If $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \cdots \geq \lambda_n$
then $H^0(G/B, \mathcal{L}_\lambda)$ is the irreducible $GL_n(\mathbb{C})$-module with a vector v_λ such that $B \cdot (\mathbb{C}\text{-span}\, v_\lambda) = \mathbb{C}\text{-span}\, v_\lambda$
If $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \cdots \geq \lambda_n$ is not satisfied
then $H^0(G/B, \mathcal{L}_\lambda) = 0$

(c) If $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_1 \geq \cdots \geq \lambda_n$ then
$$\dim(H^0(G/B, \mathcal{L}_\lambda)) = \prod_{i<j} \frac{(\lambda_i - i) - (\lambda_j - j)}{(j-i)}$$

<u>Theorem</u> Fl_μ where $\mu = (\mu_1, \dots, \mu_r)$ is a composition of n.

For $(m_1, \dots, m_r) \in \mathbb{Z}^r$ define

$$\lambda = (\underbrace{m_1, \dots, m_1}_{\mu_1}, \underbrace{m_2, \dots m_2}_{\mu_2}, \dots, \underbrace{m_r, \dots, m_r}_{\mu_r})$$

and a group homomorphism $X^\lambda : P_\mu \to \mathbb{C}^\times$ by

$$X^\lambda \begin{pmatrix} D_1 & & & * \\ & D_2 & & \\ & & \ddots & \\ 0 & & & D_r \end{pmatrix} = \det(D_1)^{m_1} \cdots \det(D_r)^{m_r}.$$

Let $\mathbb{C}_\lambda = \mathbb{C}\text{-span}\{v_\lambda\}$ be the 1-dimensional P_μ-module given by

$$b v_\lambda = X^\lambda(b) v_\lambda, \quad \text{for } b \in P_\mu.$$

<u>Then</u> (a) $\text{Pic}(Fl_\mu) \cong \mathbb{Z}^r$

$$\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda \longleftarrow (m_1, \dots, m_r)$$

(b) If $m_1 \geq m_2 \geq \cdots \geq m_r$ then

$$H^0(Fl_\mu, \mathcal{L}_\lambda) \cong L(\lambda)$$

where $L(\lambda)$ is the irreducible $GL_n(\mathbb{C})$-module with a vector v_λ such that

$$b v_\lambda = X^\lambda(b) v_\lambda, \quad \text{for } b \in P_\mu$$

If $m = (m_1, \dots, m_r)$ does not satisfy $m_1 \geq \cdots \geq m_r$ then $H^0(Fl_\mu, \mathcal{L}_\lambda) = 0$.

The case of $\mathbb{P}^1$

$$\mathbb{P}^1 = \left\{ 0 \subseteq V_1 \subseteq \mathbb{C}^2 \;\middle|\; \begin{array}{l} V_1 \text{ is a } \mathbb{C}\text{-submodule} \\ \dim V_1 = 1 \end{array} \right\}$$

The favourite line is

$E_1 = \text{span}\{e_1\}$ with $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then

$$\text{Stab}_G(E_1) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{C}) \right\} = B$$

Use

$$\begin{aligned} \varphi : G/B &\longrightarrow \mathbb{P}^1 \\ gB &\longmapsto gE_1 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} B &\longmapsto \mathbb{C}\text{-span}\left\{ \begin{pmatrix} a \\ c \end{pmatrix} \right\} = [a, c] \end{aligned}$$

to identify G/B and $\mathbb{P}^1$.

Because $S_2 = \{1, X\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} = \{1, s_1\}$

$$G = \bigsqcup_{w \in S_2} BwB = B \sqcup Bs_1B.$$

Then
$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B = [1, 0]$$

$$Bs_1B = \{ y_1(c) B \mid c \in \mathbb{C} \} = \left\{ \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} B \;\middle|\; c \in \mathbb{C} \right\}$$

$$= \{ [c, 1] \mid c \in \mathbb{C} \} = U_1.$$

Let $B^- = s_1 B s_1 = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{C}) \right\}$

$= \left\{ \begin{pmatrix} d & 0 \\ b & a \end{pmatrix} \in GL_2(\mathbb{C}) \right\} = \left\{ \begin{matrix} \text{lower triangular} \\ \text{matrices in } GL_2(\mathbb{C}) \end{matrix} \right\}$

Then

$$\begin{pmatrix} c & 0 \\ b & a \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ bc^{-1} & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ bc^{-1} & 1 \end{pmatrix} B$$

or, equivalently, $[c, b] = [1, bc^{-1}]$ in $\mathbb{P}^1$.

If $z \neq 0$ then

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} B = \begin{pmatrix} z^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & 1 \\ 0 & -z^{-1} \end{pmatrix} B = \begin{pmatrix} z^{-1} & 1 \\ 1 & 0 \end{pmatrix} B$$

or, equivalently, $[1, z] = [z^{-1}, 1]$ in $\mathbb{P}^1$.

$$\mathcal{U}_0 = \{[1, z] \mid z \in \mathbb{C}\} = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} B \;\middle|\; z \in \mathbb{C} \right\}$$

$$= B^- B$$

$$\mathcal{U}_1 = \{[z, 1] \mid z \in \mathbb{C}\} = \left\{ \begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix} B \;\middle|\; z \in \mathbb{C} \right\}$$

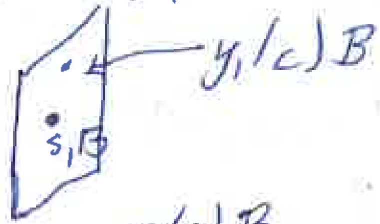
$$= B s_1 B = s_1 (s_1 B s_1) B = s_1 B^- B.$$

Then $\mathcal{U}_0 \cap \mathcal{U}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} B \;\middle|\; z \in \mathbb{C}^\times \right\} = \left\{ \begin{pmatrix} z^{-1} & 1 \\ 1 & 0 \end{pmatrix} B \;\middle|\; z \in \mathbb{C}^\times \right\}$

since $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} B = \begin{pmatrix} z^{-1} & 1 \\ 1 & 0 \end{pmatrix} B$.

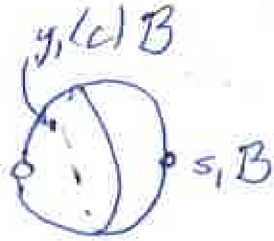
Alg. Geom Week 4 ⑧ The case of $\mathbb{P}^1$ ③

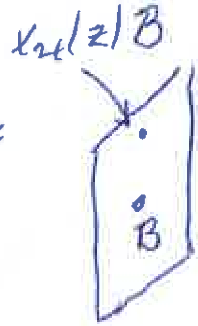
Let $x_{21}(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ and $y_1(c) = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$.

$\mathbb{P}^1 = B \sqcup Bs_1B = $ B $\sqcup$ 

$= B \sqcup U_1$

$=$ B $\sqcup$ 

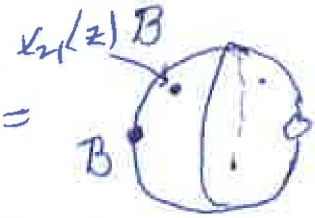
$=$ B $\sqcup$ 

$\mathbb{P}^1 = B^-B \sqcup s_1B =$ 

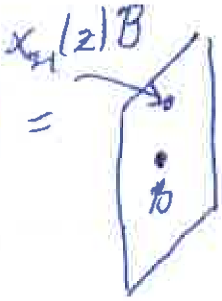
$\sqcup$ s_1B

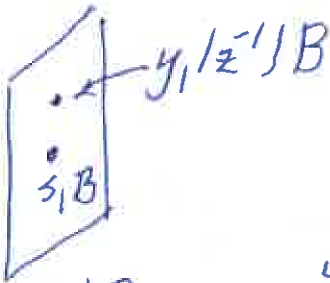
$= U_0 \sqcup s_1B$

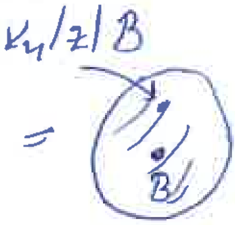
$=$ 

$\sqcup$ s_1B $=$ 

$\sqcup$ s_1B

$\mathbb{P}^1 = U_0 \cup U_1 =$ 

$\cup$ 

$=$ 

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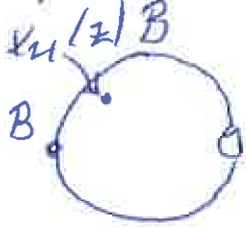
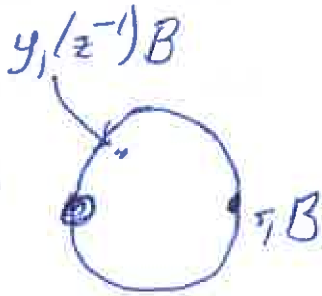
$=$ 

$\cup$ 

$x_{21}(z)B = y_1(z^{-1})B$

$=$ 

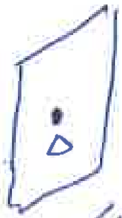
$\mathbb{P}^1$ as a scheme

$$\mathbb{P}^1 = U_0 \cup U_1 \quad \text{with} \quad U_0 \simeq \mathbb{C}$$
$$U_1 \simeq \mathbb{C}$$

and $U_0 \cap U_1 \simeq \mathbb{C}^\times = \mathbb{C} - \{0\}$

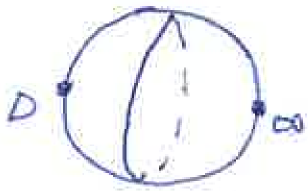
Then

$$U_0 \simeq \mathbb{C} \simeq \operatorname{Spec}(\mathbb{C}[z])$$
$$U_1 \simeq \mathbb{C} \simeq \operatorname{Spec}(\mathbb{C}[z^{-1}])$$

$U_0 =$ 

$U_1 =$ 

$$U_0 \cap U_1 \simeq \mathbb{C}^\times \simeq \operatorname{Spec}(\mathbb{C}[z, z^{-1}])$$

and $\mathbb{P}^1 = U_0 \cup U_1 =$ 

$$\operatorname{res}^{U_0}_{U_0 \cap U_1} : \mathbb{C}[z] \longrightarrow \mathbb{C}[z, z^{-1}]$$
$$f(z) \longmapsto f(z)$$

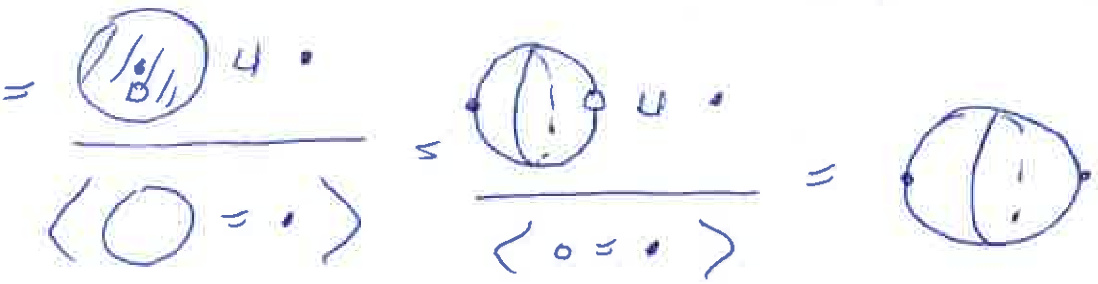
$$\operatorname{res}^{U_1}_{U_0 \cap U_1} : \mathbb{C}[z^{-1}] \longrightarrow \mathbb{C}[z, z^{-1}]$$
$$g(z^{-1}) \longmapsto g(z^{-1}).$$

$\mathbb{P}^1$ as a CW-complex

Let $X_0 = pt$ and

$$S^1 \xrightarrow{g} X_0$$
$$(x_1, x_2) \longmapsto \bullet$$

Then
$$D^2 \sqcup_g pt = D^2 \sqcup_g X_0 = \frac{D^2 \sqcup pt}{\langle (x_1,x_2) = \bullet \mid (x_1,x_2) \in S^1 \rangle}$$

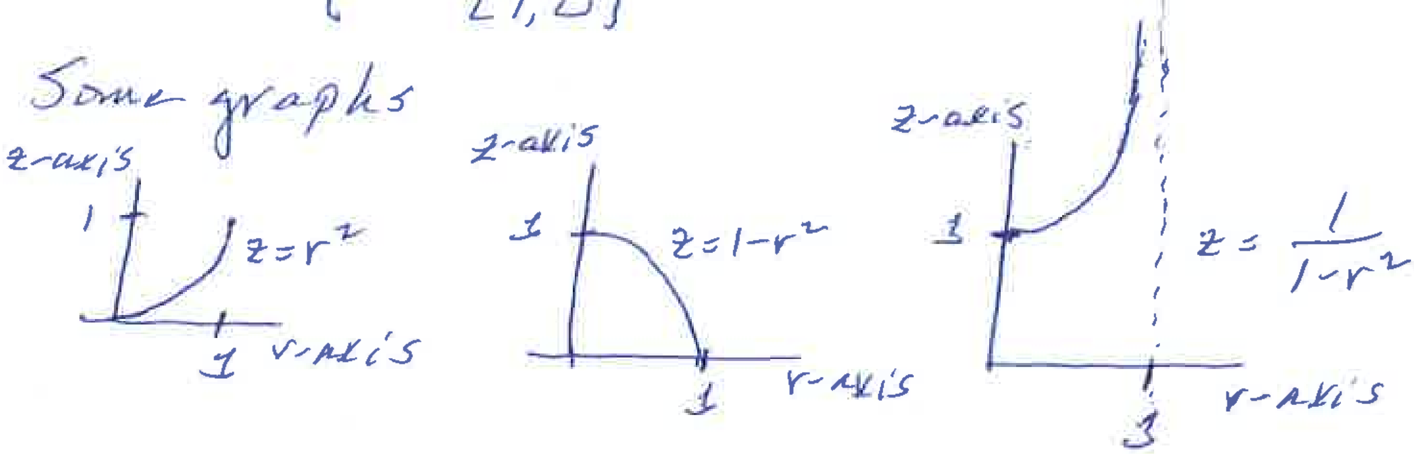
$=$ 

Define
$$D^2 \xrightarrow{f} \mathbb{P}^1$$
$$(x_1,x_2) \longmapsto [x_1 + i x_2, \sqrt{1-(x_1^2+x_2^2)}\,]$$

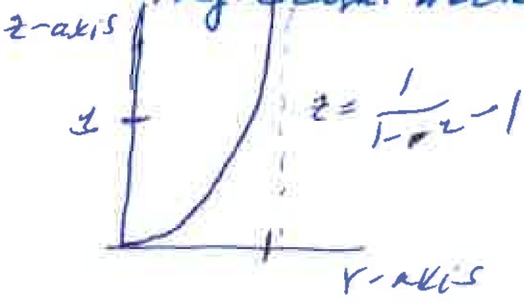
Writing $x_1 + i x_2 = r e^{i\theta}$ with $r^2 = x_1^2 + x_2^2$ then

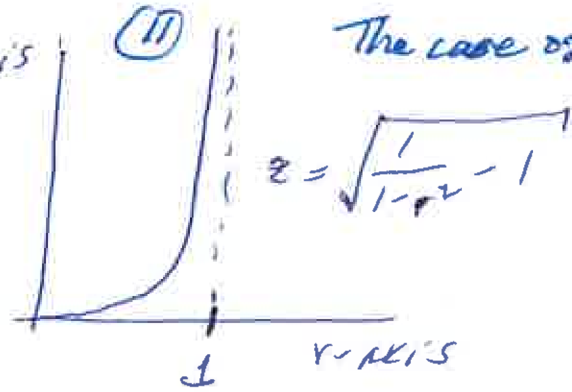
$$[x_1 + i x_2, \sqrt{1-(x_1^2+x_2^2)}\,] = [r e^{i\theta}, \sqrt{1-r^2}\,]$$
$$= \begin{cases} \left[\frac{r}{\sqrt{1-r^2}} e^{i\theta}, 1\right], & \text{if } r \neq 1 \\ [1, 0] & \end{cases}$$

Some graphs



Alg. Geom. Week 4 (11) The case of $\mathbb{P}^1$. (6)





So $\sqrt{\frac{1}{1-r^2} - 1} = \sqrt{\frac{1-(1-r^2)}{1-r^2}} = \frac{r}{\sqrt{1-r^2}}$ for $r \in \mathbb{R}_{[0,1]}$

then $f : D^2 \longrightarrow \mathbb{P}^1$ given by

$$f(x_1, x_2) = \begin{cases} \left[\frac{r}{\sqrt{1-r^2}} e^{i\theta}, 1\right], & \text{if } r \neq 1 \\ [1, 0], & \text{if } r = 1 \end{cases}$$

(where $re^{i\theta} = x_1 + ix_2$) is surjective and

$$\mathrm{res}^{D^2}_{S^1} f : S^1 \longrightarrow \mathbb{P}^1$$
$$(x_1, x_2) \longmapsto [1, 0].$$

So $\mathbb{P}^1 \cong D^2 \sqcup_g pt$ where $[1,0]$ corresponds to pt.

Alg. Geom. Week 4 (12) 18.08.2018 UniMelb A. Ram (1)

$\underline{\mathrm{Spec}(\mathbb{C}[x])} = (X, \mathcal{T}_X, \mathcal{O}_X)$

$X = \{\text{prime ideals in } \mathbb{C}[x]\}$

The ring $\mathbb{C}[x]$ is a PID, i.e. every ideal is generated by one element (the proof is via the Euclidean algorithm).

If $f \in \mathbb{C}[x]$ and $I = f\mathbb{C}[x]$ is a prime ideal then f is irreducible (If $f\mathbb{C}[x] = I$ is prime and $f = ab$ then $a \in I$ or $b \in I$ so that $I = a\mathbb{C}[x]$ or $I = b\mathbb{C}[x]$).

Since $\mathbb{C}$ is algebraically closed and f is irreducible then ~~there~~ $f = 0$ ~~or $f=1$~~ or there exists $\alpha \in \mathbb{C}$ with $f = x - \alpha$.

So $X = \{(x-\alpha)\mathbb{C}[x] \mid \alpha \in \mathbb{C}\} \cup \{0\}$

Identify X with $\mathbb{C} \cup \{*\}$ via the bijection

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \mathbb{C} \cup \{*\} \\ (x-\alpha)\mathbb{C}[x] & \longmapsto & \alpha. \\ 0 & \longmapsto & *. \end{array}$$

(Why don't we consider the case when $f = 1$? Why is $\mathbb{C}[x]$ not a prime ideal in $\mathbb{C}[x]$?)

Let $S \subseteq \mathbb{C}[x]$. Assume $S \neq \emptyset$.

Since $\mathbb{C}[x]$ is a PID the ideal $\langle S \rangle$ generated by S is generated by one element

$$\langle S \rangle = g\,\mathbb{C}[x] \quad \text{where } g = \gcd(S).$$

Then

$$V(S) = \{p \in X \mid \text{If } g \in S \text{ then } g = 0 \text{ in } \tfrac{\mathbb{C}[x]}{p}\}$$

$$= \{p \in X \mid \text{if } g \in S \text{ then } g \in p\}$$

$$= \{p \in X \mid S \subseteq p\} = \{p \in X \mid \langle S \rangle \subseteq p\}$$

$$= \{p \in X \mid g\,\mathbb{C}[x] \subseteq p\} = \{p \in X \mid g \in p\}$$

$$= V(\{g\}).$$

Case 1: $g = 0$. Then

$$V(\{0\}) = \{p \in X \mid 0 \in p\} = X = \mathbb{C} \sqcup \{*\}$$

Case 2: $g = 1$. Then

$$V(\{1\}) = V(\mathbb{C}[x]) = \{p \in X \mid \mathbb{C}[x] \subseteq p\} = \emptyset$$

Case 3: $g \neq 0$ and $g \neq 1$. Let $g = (x-\alpha_1)^{m_1} \cdots (x-\alpha_r)^{m_r}$ be the factorization of g. Using that $g \notin \{0\}$,

$$V(\{g\}) = \{p \in X \mid g \in p\} = \{\alpha \in \mathbb{C} \mid g \in (x-\alpha)\mathbb{C}[x]\}$$

$$= \{\alpha \in \mathbb{C} \mid x-\alpha \text{ divides } g\}$$

$$= \{\alpha_1, \dots, \alpha_r\}.$$

So, the closed sets of $(X, \mathcal{T}_X)$ are

$\emptyset$ and $X = \mathbb{C} \sqcup \{*\}$ and finite subsets of $\mathbb{C}$.

Let $g \in \mathbb{C}[x]$.

<u>Case 1</u>: $g = 0$. Then

$$X_0 = \{p \in X \mid 0 \notin p\} = \emptyset$$

<u>Case 2</u>: $g = 1$. Then

$$X_1 = \{p \in X \mid 1 \notin p\} = X = \mathbb{C} \sqcup \{*\}.$$

<u>Case 3</u>: $g \neq 0$ and $g \neq 1$. Assume g is monic and let

$$g = (x-\alpha_1)^{m_1} \cdots (x-\alpha_r)^{m_r}, \text{ with } \alpha_1, \ldots, \alpha_r \text{ distinct,}$$

be the factorization of g. Then

$$X_g = \{p \in X \mid g \notin p\} = \{\alpha \in \mathbb{C} \mid g \notin (x-\alpha)\mathbb{C}[x]\} \cup \{*\}$$

$$= \{\alpha \in \mathbb{C} \mid (x-\alpha) \text{ is not a factor of } g\} \cup \{*\}$$

$$= \{\alpha \in \mathbb{C} \mid g(\alpha) \neq 0\} \cup \{*\}$$

$$= X - \{\alpha_1, \ldots, \alpha_r\} = V(\{g\})^c = V(\{\tilde{g}\})^c$$

where $\tilde{g} = (x-\alpha_1) \cdots (x-\alpha_r)$.

So $\mathcal{T}_X = \{X_g \mid g \in \mathbb{C}[x]\}$

and every open set is a basic set X_g.

Determination of $\mathcal{O}_X(X_g)$

Case 1: $g=0$. Then

$$\mathcal{O}_X(X_0) = \mathcal{O}_X(\phi) = 1$$

Case 2: $g=1$. Then

$$\mathcal{O}_X(X_1) = \mathcal{O}_X(X) = \mathbb{C}[x].$$

Case 3 $\alpha_1, \dots \alpha_r \in \mathbb{C}$ are distinct and

$$g = (x-\alpha_1)\cdots(x-\alpha_r).$$

Then

$$\mathcal{O}_X(X_g) = \mathbb{C}[x][\tfrac{1}{g}] = \left\{ \frac{f}{g^k} \,\middle|\, k \in \mathbb{Z} \text{ and } f \in \mathbb{C}[x] \right\}$$

$$= \left\{ \frac{f}{q} \,\middle|\, \begin{array}{l} f, q \in \mathbb{C}[x], \text{ if } \alpha \in \mathbb{C} - \{\alpha_1, \dots, \alpha_r\} \\ \text{then } x-\alpha \text{ does not divide } q \end{array} \right\}$$

$$= \left\{ \frac{f}{q} \,\middle|\, \begin{array}{l} f, q \in \mathbb{C}[x] \text{ and if } \alpha \in \mathbb{C} - \{\alpha_1, \dots, \alpha_r\} \\ \text{then } q(\alpha) \neq 0 \end{array} \right\}$$

$$= \{\text{regular functions on } \mathbb{C} - \{\alpha_1, \dots, \alpha_r\}\}.$$