Flag varieties

Let $n \in \mathbb{Z}_{>1}$.

A <u>composition of n</u> is a sequence $\mu = (\mu_1, \ldots, \mu_r)$ with $\mu_1, \ldots, \mu_r \in \mathbb{Z}_{>0}$ such that

$$\mu_1 + \cdots + \mu_r = n.$$

- <u>Projective space</u> $\mathbb{P}^{n-1}$ is the space of lines in $\mathbb{C}^n$,

$$\mathbb{P}^{n-1} = \left\{ 0 \subseteq V_1 \subseteq \mathbb{C}^n \;\middle|\; \begin{array}{l} V_1 \text{ is a } \mathbb{C}\text{-subspace of } \mathbb{C}^n \\ \dim_{\mathbb{C}} V_1 = 1 \end{array} \right\}$$

- Let $k \in \{1, \ldots, n\}$. The <u>Grassmannian</u> $Gr_k(n)$ is the space of k-planes in $\mathbb{C}^n$,

$$Gr_k(n) = \left\{ 0 \subseteq V_k \subseteq \mathbb{C}^n \;\middle|\; \begin{array}{l} V_k \text{ is a } \mathbb{C}\text{-subspace of } \mathbb{C}^n \\ \dim_{\mathbb{C}} V_k = k \end{array} \right\}$$

- The <u>flag variety</u> Fl is the space of flags in $\mathbb{C}^n$,

$$Fl = \left\{ (0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n) \;\middle|\; \begin{array}{l} V_j \text{ is a } \mathbb{C}\text{-submodule of } \mathbb{C}^n \\ \dim_{\mathbb{C}} V_j = j \end{array} \right\}$$

- Let $\mu = (\mu_1, \ldots, \mu_r)$ be a composition of n. The <u>μ-partial flag variety</u> Fl_μ is the space of μ-partial flags in $\mathbb{C}^n$,

$$Fl_\mu = \left\{ (0 \subseteq V_{\mu_1} \subseteq V_{\mu_1+\mu_2} \subseteq \cdots \subseteq V_{\mu_1+\cdots+\mu_{r-1}} \subseteq \mathbb{C}^n) \;\middle|\; \begin{array}{l} V_{\mu_j} \text{ is a } \mathbb{C}\text{-submodule of } \mathbb{C}^n \text{ and} \\ \dim V_{\mu_1+\cdots+\mu_i} = \mu_1+\cdots+\mu_i \end{array} \right.$$

Flag varieties as homogeneous spaces G/P

The group
$$G = GL_n(\mathbb{C}) \quad \text{acts on} \quad \mathbb{C}^n$$
by matrix multiplication.

Let $S \subseteq \mathbb{C}^n$. The <u>stabilizer of S</u> is
$$\mathrm{Stab}_G(S) = \{ g \in G \mid gS = S \},$$
where $gS = \{ gs \mid s \in S \}$.

Let
$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} i\text{th} \qquad \text{for } i \in \{1, \ldots, n\}.$$

Then $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{C}^n$.

- The favourite line in $\mathbb{C}^n$ is $E_1 = \mathbb{C}\text{-span}\{e_1\}$
- The favorite k-plane in $\mathbb{C}^n$ is $E_k = \mathbb{C}\text{-span}\{e_1, \ldots, e_k\}$
- The favourite flag in $\mathbb{C}^n$ is
$$E = (0 \subseteq E_1 \subseteq \cdots \subseteq E_{n-1} \subseteq \mathbb{C}^n)$$
- Let μ be a composition of n, $\mu = (\mu_1, \ldots, \mu_r)$. The favorite μ-partial flag in $\mathbb{C}^n$ is
$$E_\mu = (0 \subseteq E_{\mu_1} \subseteq E_{\mu_1+\mu_2} \subseteq \cdots \subseteq E_{\mu_1+\cdots+\mu_{r-1}} \subseteq \mathbb{C}^n).$$

Let $\mu = (\mu_1, \ldots, \mu_r)$ be a composition of n.

$$P_\mu = \left\{ \begin{pmatrix} \boxed{\mu_1} & & & * \\ & \boxed{\mu_2} & & \\ & & \ddots & \\ & & & \boxed{\mu_{r-1}} & \\ 0 & & & & \boxed{\mu_r} \end{pmatrix} \right\} = \left\{ \begin{array}{l} (\mu_1, \ldots, \mu_r)\text{-block upper} \\ \text{triangular matrices} \\ \text{in } GL_n(\mathbb{C}) \end{array} \right\}$$

Let

$$P_1 = P_{1,n-1}, \quad P_k = P_{k,n-k}, \quad B = P_{1,1,\ldots,1}$$

Theorem

(a) Projective space $\mathbb{P}^{n-1}$

$\mathrm{Stab}_G(E_1) = P_1$ and
$$\begin{array}{ccc} G/P_1 & \xrightarrow{\sim} & \mathbb{P}^{n-1} \\ gP_1 & \longmapsto & gE_1 \end{array}$$

(b) Grassmannians $Gr_k(n)$

$\mathrm{Stab}_G(E_k) = P_k$ and
$$\begin{array}{ccc} G/P_k & \xrightarrow{\sim} & Gr_k(n) \\ gP_k & \longmapsto & gE_k \end{array}$$

(c) The flag variety Fl

$\mathrm{Stab}_G(E) = B$ and
$$\begin{array}{ccc} G/B & \xrightarrow{\sim} & Fl \\ gB & \longmapsto & gE \end{array}$$

(d) partial flag varieties Fl_μ.

$\mathrm{Stab}_G(E_\mu) = P_\mu$ and
$$\begin{array}{ccc} G/P_\mu & \xrightarrow{\sim} & Fl_\mu \\ gP_\mu & \longmapsto & gE_\mu. \end{array}$$

<u>Permutations</u> Let $n \in \mathbb{Z}_{\geq 1}$.

Let E_{ij} denote the $n\times n$ matrix with 1 in the (i,j) entry and 0 elsewhere.

A <u>permutation</u> of n is an $n\times n$ matrix w such that

(a) there is exactly one nonzero entry in each row and each column

(b) the nonzero entries are 1.

The <u>symmetric group S_n</u> is

$$S_n = \{\text{permutations of } n\}$$

with operation matrix multiplication.

Let

$$s_i = \begin{array}{c c} & \begin{array}{cc} i & i+1 \end{array} \\ \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} \end{array} = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i},$$

for $i \in \{1, \ldots, n\}$.

Let $w \in S_n$. A <u>reduced word for w</u> is a product

$$w = s_{i_1} \cdots s_{i_\ell} \quad \text{with } \ell \text{ minimal}.$$

The <u>length of w</u> is the minimal ℓ such that there is a product $w = s_{i_1} \cdots s_{i_\ell}$.

Let $w \in S_n$. Identify w with a bijection

$$w: \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$$
$$i \longmapsto w(i)$$

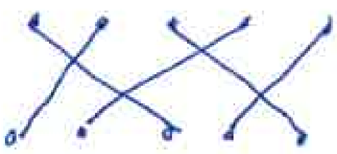
by the formula

$$w = E_{w(1),1} + \cdots + E_{w(n),n}.$$

Identify w with a graph with
- n dots in the top row,
- n dots in the bottom row,
- edges $i \to w(i)$ connecting the ith dot of the top row to the $w(i)$th dot of the bottom row.

<u>Example</u>

$$w = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{array}{c} \{1,2,3,4,5\} \longrightarrow \{1,2,3,4,5\} \\ 1 \longmapsto 3 \\ 2 \longmapsto 1 \\ 3 \longmapsto 5 \\ 4 \longmapsto 2 \\ 5 \longmapsto 4 \end{array}$$

$\longleftrightarrow$ 

Theorem F1

(a) $$G = \bigsqcup_{w \in S_n} BwB$$

(b) If $w \in S_n$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w then

$$BwB = \{ y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) B \mid c_1, \ldots, c_\ell \in \mathbb{C} \}$$

(c) If $w \in S_n$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word and

$$c_1, \ldots, c_\ell \in \mathbb{C} \quad \text{and} \quad c_1', \ldots, c_\ell' \in \mathbb{C}$$

and $$y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) B = y_{i_1}(c_1') \cdots y_{i_\ell}(c_\ell') B$$

then $$c_1 = c_1', \; c_2 = c_2', \; \ldots, \; c_\ell = c_\ell'.$$

For $i \in \{1, 2, \ldots, n\}$ and $c \in \mathbb{C}$ let

$$y_i(c) = \begin{array}{c c} & \begin{array}{cc} i & i+1 \end{array} \\ \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array} & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & c \;\; 1 & \\ & & & 1 \;\; 0 & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array} = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}$$

Theorem $\mathbb{P}^{n-1}$

(a)
$$G = \bigsqcup_{w \in S_n/S_{n-1}} BwP_1$$

(b) If $w \in S_n$ is the minimal length element of the coset wS_{n-1} and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w then

$$BwP_1 = \{ y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) P_1 \mid c_1, \ldots, c_\ell \in \mathbb{C} \}$$

(c) If $w \in S_n$ is the minimal length element of the coset wS_{n-1} and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w and $c_1, \ldots, c_\ell \in \mathbb{C}$ and $c_1', \ldots, c_\ell' \in \mathbb{C}$

and
$$y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) P_1 = y_{i_1}(c_1') \cdots y_{i_\ell}(c_\ell') P_1$$
then
$$c_1 = c_1', \ c_2 = c_2', \ \ldots, \ c_\ell = c_\ell'.$$

Theorem $Gr_k(a)$

(a)
$$G = \bigsqcup_{w \in S_n/S_k \times S_{n-k}} B w P_k$$

(b) If $w \in S_n$ is the minimal length element of the coset $w(S_k \times S_{n-k})$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w then

$$B w P_k = \{ y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) P_k \mid c_1, \ldots, c_\ell \in \mathbb{C} \}$$

(c) If $w \in S_n$ is the minimal length element of the coset $w(S_k \times S_{n-k})$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w ~~then~~ and $c_1, \ldots, c_\ell \in \mathbb{C}$ and $c_1', \ldots, c_\ell' \in \mathbb{C}$ and $y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) P_k = y_{i_1}(c_1') \cdots y_{i_\ell}(c_\ell') P_k$ then $c_1 = c_1', c_2 = c_2', \ldots, c_\ell = c_\ell'$.

Theorem F1μ Let $\mu=(\mu_1,\ldots,\mu_r)$ be a composition of n. Let

$$S_\mu = S_{\mu_1}\times\cdots\times S_{\mu_r}$$

(a)
$$G = \bigsqcup_{w\in S_n/S_\mu} BwP_\mu$$

(b) If $w\in S_n$ is the minimal length element of the coset wS_μ and

$w=s_{i_1}\cdots s_{i_\ell}$ is a reduced word for w then

$$BwP_\mu=\{y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)P_\mu \mid c_1,\ldots,c_\ell\in\mathbb{C}\}.$$

(c) If $w\in S_n$ is the minimal length element of the coset wS_μ and

$w=s_{i_1}\cdots s_{i_\ell}$ is a reduced word for w ~~then~~ and

$c_1,\ldots,c_\ell\in\mathbb{C}$ and $c_1',\ldots,c_\ell'\in\mathbb{C}$ and

$$y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)P_\mu = y_{i_1}(c_1')\cdots y_{i_\ell}(c_\ell')P_\mu$$

then
$$c_1=c_1',\ c_2=c_2',\ \ldots,\ c_\ell=c_\ell'.$$