Algebraic geometry Week 2

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Recall: A ringed space is a pair

$$(X, \mathcal{O}_X)$$

topological space — sheaf of rings on X (commutative, unital) "structure sheaf"

Example 1 (Affine space as a ringed space).

k an algebraically closed field.

$\mathbb{A}^n = \mathbb{A}^n_k = k^n$ "affine n-space (over k)"

Let $S \subseteq k[x_1, \dots x_n]$. Define

$$V(S) = \left\{ (c_1, \dots, c_n) \in \mathbb{A}^n \;\middle|\; \begin{array}{l} \text{if } f \in S \text{ then} \\ f(c_1, \dots, c_n) = 0 \end{array} \right\} \subseteq \mathbb{A}^n$$

$V(S)$ is an "affine algebraic set".

For example:

$\mathbb{A}^n = V(\{0\})$, $\quad \phi = V(\{1\})$ and

$\{(c_1, \dots, c_n)\} = V(\{x_1 - c_1, \dots, x_n - c_n\})$

Zariski topology

$U \subseteq \mathbb{A}^n$ is open if there exists

$S \subseteq k[x_1, \dots, x_n]$ such that $U = \mathbb{A}^n - V(S)$.

HW: Show that this defines a topology on $\mathbb{A}^n$

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Structure sheaf $\mathcal{O}_{\mathbb{A}^n}$

Let $U \subseteq \mathbb{A}^n$ be open. A <u>regular function on U</u> is a function $\varphi : U \to k$ such that

if $p \in U$ then there exists $U_p \subseteq U$ with U_p open and $p \in U_p$ and $f, g \in k[x_1, \ldots, x_n]$ such that

if $q \in U_p$ then $f(q) \neq 0$ and $\varphi(q) = \dfrac{g(q)}{f(q)}$.

Let

$$\mathcal{O}_{\mathbb{A}^n}(U) = \{ \varphi : U \to k \mid \varphi \text{ is a regular function on } U \}.$$

HW: Show that $\mathcal{O}_{\mathbb{A}^n}$ is a sheaf of rings on $\mathbb{A}^n$ (with the Zariski topology).

<u>Proposition</u> $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) = k[x_1, \ldots, x_n]$.

<u>HW</u>: Show that if k is not algebraically closed then $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n)$ is not always $k[x_1, \ldots, x_n]$. (Hint: Use $k = \mathbb{R}$ and $\varphi(x) = \frac{1}{x^2+1}$).

Example 2 (Projective space as a ringed space)

$$\begin{tikzcd}[column sep=small]
 & \mathbb{A}^{n+1} \arrow[d, phantom, "\cup" rotate=90] & \\
 & \mathbb{A}^{n+1}-\{0\} \arrow[dl] \arrow[dr] & \\
\mathbb{P}^n_k = \dfrac{\mathbb{A}^{n+1}-\{0\}}{\sim} & & \left\{ \begin{array}{l} \text{lines through the} \\ \text{origin in } k^{n+1} \end{array} \right\}
\end{tikzcd}$$

$$(\lambda_0, \dots, \lambda_n) \mapsto [\lambda_0 : \dots : \lambda_n] \qquad (\lambda_0, \dots, \lambda_n) \mapsto L_{(\lambda_0, \dots, \lambda_n)}$$

$\mathbb{A}^{n+1}-\{0\}$ has the subspace topology, i.e.

$\mathcal{U}$ is open if there exists V open in $\mathbb{A}^{n+1}$ such that $\mathcal{U} = V \cap (\mathbb{A}^{n+1}-\{0\})$

Define

$(\lambda_0, \dots, \lambda_n) \sim (\mu_0, \dots, \mu_n)$ if there exists $c \in k^\times$ such that $c(\lambda_0, \dots, \lambda_n) = (\mu_0, \dots, \mu_n)$.

Excursion (complex projective space).

Let $\tilde{\mathbb{P}}^n_{\mathbb{C}}$ be $\dfrac{\mathbb{C}^{n+1}-\{0\}}{\sim}$ with the quotient topology coming from $\mathbb{C}^{n+1}-\{0\}$ with the standard topology.

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HW (a) Show that $\mathbb{P}^n_{\mathbb{C}}$ is quasicompact and Hausdorff.

(b) Show that $\mathbb{P}^n_k$ is quasicompact and not Hausdorff.

Zariski topology on $\mathbb{P}^n_k$ Let $n \in \mathbb{Z}_{>0}$.

Let $S \subseteq k[x_0, \dots x_n]$ be a set of homogeneous polynomials. Define

$$V_{\mathbb{P}}(S) = \left\{ [\lambda_0, \dots, \lambda_n] \in \mathbb{P}^{n+1}_k \;\middle|\; \begin{array}{l} \text{if } f \in S \text{ then} \\ f(\lambda_0, \dots, \lambda_n) = 0 \end{array} \right\}.$$

The $V_{\mathbb{P}}(S)$ are "projective algebraic sets".

HW: Show that $V_{\mathbb{P}}(S)$ is well defined, i.e. if $[\lambda_0, \dots, \lambda_n] \in V_{\mathbb{P}}(S)$ and $[\lambda_0, \dots, \lambda_n] = [\mu_0, \dots, \mu_n]$ then $[\mu_0, \dots, \mu_n] \in V_{\mathbb{P}}(S)$.

Define $U \subseteq \mathbb{P}^n_k$ to be open if
there exists a set of homogeneous polynomials $S \subseteq k[x_1, \dots, x_n]$ such that $U = \mathbb{P}^n_k - V_{\mathbb{P}}(S)$.

HW: Show that this agrees with the quotient topology coming from $\mathbb{A}^{n+1}_k - \{0\}$.

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Structure sheaf on $\mathbb{P}_k^n$

Let $U \subseteq \mathbb{P}_k^n$ be open. A <u>regular function on U</u> is a function $\varphi: U \to k$ such that

if $a \in U$ then there exists $U_a \subseteq U$ with U_a open and $a \in U_a$ and $f, g \in k[x_1, \ldots, x_n]$ homogeneous such that

if $x \in U_a$ then $f(x) \neq 0$ and $\varphi(x) = \dfrac{g(x)}{f(x)}$.

Define

$$\mathcal{O}_{\mathbb{P}^n}(U) = \{\text{regular functions on } U\}.$$

<u>HW</u>: Show that $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf of rings on $\mathbb{P}_k^n$.

<u>Proposition</u> $\mathcal{O}_{\mathbb{P}_k^n}(\mathbb{P}^n) = k$.

<u>HW</u> (a) Prove this for $\mathbb{P}_{\mathbb{C}}^n$.

(b) Prove this for $\mathbb{P}_k^n$.

Define a map

$$\mathbb{P}^{k-1}_{\mathbb{C}} \xhookrightarrow{\iota_{k-1}} \mathbb{P}^{k}_{\mathbb{C}}$$
$$[\lambda_0, \ldots, \lambda_{k-1}] \longmapsto [\lambda_0, \ldots, \lambda_{k-1} : 0].$$

(This comes from $\mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}$
$(\lambda_0, \ldots, \lambda_{k-1}) \mapsto (\lambda_0, \ldots, \lambda_{k-1}, 0)$)

Let $D^{2k} = \{(y_1, \ldots, y_{2k}) \in \mathbb{R}^{2k} \mid y_1^2 + \cdots + y_{2k}^2 \le 1\}$

and define

$$f_k : D^{2k} \longrightarrow \mathbb{P}^k_{\mathbb{C}} \quad \text{by}$$

$$f_k(y_1, \ldots, y_{2k}) = [y_1 + iy_2, \ldots, y_{2k-1} + iy_{2k}, \sqrt{1 - \|y\|^2}]$$

Prove that f_k is continuous and surjective.

Let $S^{2k-1} = \{(y_1, \ldots, y_{2k}) \in \mathbb{R}^{2k} \mid y_1^2 + \cdots + y_{2k}^2 = 1\}$

Let $g_k : S^{2k-1} \longrightarrow \mathbb{P}^k_{\mathbb{C}}$ be the restriction of f_k to S^{2k-1}

Show that $\mathrm{im}(g_k) = \mathbb{P}^{k-1}_{\mathbb{C}}$ (contained in $\mathbb{P}^k_{\mathbb{C}}$).

Define $D^{2k} \sqcup \mathbb{P}^{k-1}_{\mathbb{C}} \xrightarrow{\varphi} ((\mathbb{P}^k_{\mathbb{C}} - \mathbb{P}^{k-1}_{\mathbb{C}}) \cup \mathbb{P}^{k-1}_{\mathbb{C}}) = \mathbb{P}^k_{\mathbb{C}}$

by $\varphi(d) = f_k(d)$, if $d \in D^{2k}$, and

$\varphi(x) = \iota_{k-1}(x)$, if $x \in \mathbb{P}^{k-1}_{\mathbb{C}}$.

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Then, show that

$$\begin{tikzcd}
D^{2k} \sqcup \hat{\mathbb{P}}^{k-1}_{\mathbb{C}} \arrow[r] \arrow[d] & \hat{\mathbb{P}}^{k}_{\mathbb{C}} = \left(\hat{\mathbb{P}}^{k}_{\mathbb{C}} - \hat{\mathbb{P}}^{k-1}_{\mathbb{C}}\right) \cup \hat{\mathbb{P}}^{k-1}_{\mathbb{C}} \\
\dfrac{D^{2k} \sqcup \hat{\mathbb{P}}^{k-1}_{\mathbb{C}}}{\langle g_k(y) = y \mid y \in S^{2k-1} \rangle} = D^{2k} \sqcup_{g_k} \mathbb{P}^{k-1} \arrow[ur, "\simeq"', "\text{homeomorphism}"] &
\end{tikzcd}$$

Use this set up to show that $\hat{\mathbb{P}}^{k}_{\mathbb{C}}$ is a CW-complex.
Look in Hatcher for the definition of a CW-complex

<u>How are the ringed spaces</u> $(\mathbb{A}^n_k, \mathcal{O}_{\mathbb{A}^n_k})$ and $(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$ <u>related?</u>

<u>Definition</u>: A <u>prevariety</u> is a ringed space $(X, \mathcal{O}_X)$ such that there exists a finite open cover $\mathcal{S}$ such that

$$(U_i, \mathcal{O}_X|_{U_i}) \simeq (\mathbb{A}^n_k, \mathcal{O}_{\mathbb{A}^n_k}) \quad \left(\begin{array}{c}\text{as ringed}\\ \text{spaces}\end{array}\right)$$

for each $U_i \in \mathcal{S}$.

<u>Proposition</u> $(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k})$ is a prevariety.

Idea of proof:

Define

$$U_i = \{[\lambda_0, \ldots, \lambda_n] \mid \lambda_i \neq 0\}$$

$$= \{[\lambda_0, \ldots, \lambda_{i-1}, 1, \lambda_{i+1}, \ldots, \lambda_n]\} \quad \left(\text{subset of } \mathbb{P}^n_k\right)$$

Then

$$\mathbb{P}^n_k = \bigcup_{i=0}^{n} U_i \quad \text{is an open cover}$$

(prove this). Define

$$F: \mathbb{A}^n_k \longrightarrow U_0 \quad \text{by} \quad F(\lambda_1, \ldots, \lambda_n) = [1, \lambda_1, \ldots, \lambda_n]$$

and

$$F^{-1}: U_0 \longrightarrow \mathbb{A}^n_k \quad \text{by} \quad F^{-1}([\lambda_0, \lambda_1, \ldots, \lambda_n]) = \left(\frac{\lambda_1}{\lambda_0}, \ldots, \frac{\lambda_n}{\lambda_0}\right).$$

Show that F and F^{-1} are well defined, continuous and inverse to each other.

Define ring homomorphisms (isomorphisms)

$$\mathcal{O}_{\mathbb{P}^n_k}\big|_{U_i}(V) \xrightarrow{\sim} \mathcal{O}_{\mathbb{A}^n_k}(F(V))$$

$$(V \xrightarrow{f} k) \longmapsto (F(V) \xrightarrow{F^{-1}} V \xrightarrow{f} k)$$

for $V \subseteq U_i$ open.

Complete this to show $(U_i, \mathcal{O}_X|_{U_i}) \cong (\mathbb{A}^n_k, \mathcal{O}_{\mathbb{A}^n_k})$.