Algebraic Geom. Week 2 Friday   03.08.2018 UniMelb A. Ram   ①

Vector bundles and line bundles

For a sheaf $\mathcal{F}$ write

$$\mathcal{F}res^V_U = \mathcal{F}(U \hookrightarrow V) \text{ so that } \mathcal{F}res^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$$

Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space.
An <u>$\mathcal{O}_X$-module</u> is a sheaf $\mathcal{F}$ on X such that

(a) If $U \in \mathcal{T}_X$ then $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.

(b) If $U, V \in \mathcal{T}_X$ and $U \subseteq V$ then

$$\mathcal{F}res^V_U \in \mathrm{Hom}_{\mathcal{O}_X(V)}(\mathcal{F}(V), \mathcal{F}(U))$$

is an $\mathcal{O}_X(V)$-module homomorphism, where

$\mathcal{F}(U)$ is an $\mathcal{O}_X(V)$-module by

$$f_V m_U = (\mathcal{O}res^V_U f_V) \cdot m_U, \quad \text{for } m_U \in \mathcal{F}(U) \text{ and } f_V \in \mathcal{O}_X(V).$$

Let $r \in \mathbb{Z}_{>0}$. A <u>locally free sheaf of rank r</u> is an $\mathcal{O}_X$-module $\mathcal{F}$ such that there exists an open cover $\mathcal{S}$ of X and sheaf isomorphisms

$$\varphi_U : \mathcal{F}_U \to \mathcal{O}_U^{\oplus r}$$

A <u>vector bundle</u> is a locally free sheaf.
A <u>line bundle</u> is a vector bundle of rank 1.

A <u>trivial bundle</u> is a vector bundle such that

$$\mathcal{F} \cong \mathcal{O}_X^{\oplus r}.$$

Gluing (from (3.3.1) of EGA I).

Let $(X, \mathcal{T}_X)$ be a topological space.

Let $\mathcal{S}$ be an open cover of X.

For $U \in \mathcal{S}$ let

$\mathcal{F}^U$ be a sheaf on $(U, \mathcal{T}_U)$

For $U, V \in \mathcal{S}$ let

$$\theta_{UV} : (\mathcal{F}^V)_{V\cap U} \xrightarrow{\sim} (\mathcal{F}^U)_{V \cap U}$$

be an isomorphism of sheaves, such that

$$(\theta_{WV})_{U\cap V\cap W} = (\theta_{WU})_{U\cap V\cap W} \circ (\theta_{UV})_{U\cap V\cap W}.$$

Then there exists a sheaf $\mathcal{F}$ on X and sheaf isomorphisms

$$\eta_U : \mathcal{F}_U \xrightarrow{\sim} \mathcal{F}^U \quad \text{for } U \in \mathcal{S}$$

such that

if $U, V \in \mathcal{S}$ ~~and~~ then

$$\theta_{VU} = (\eta_V)_{U\cap V} \circ (\eta_U)^{-1}_{U \cap V}.$$

and $\mathcal{F}$ and the η_U are determined, up to unique isomorphism, by these conditions.

Line bundles $\mathcal{L}_{d\omega_1}$ on $\mathbb{P}^{n-1}$

Let $\mathcal{S}$ be the open cover of $(\mathbb{P}^{n-1}, \mathcal{T}^{Zar}_{\mathbb{P}^{n-1}}, \mathcal{O}_{\mathbb{P}^{n-1}})$ given by

$\mathcal{S} = \{\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}\}$, where $\mathcal{U}_i = \{[\lambda_0, \ldots, \lambda_{n-1}] \mid \lambda_i \neq 0\}$

for $i \in \{0, \ldots, n-1\}$.

Let $d \in \mathbb{Z}$.

For $i \in \{0, \ldots, n-1\}$ define

$$\mathcal{F}_i = (\mathcal{O}_{\mathbb{P}^n})_{\mathcal{U}_i}, \text{ a sheaf on } \mathcal{U}_i$$

For $i,j \in \{0, \ldots, n-1\}$ define

$$\varphi_{ij} : (\mathcal{F}_i)_{\mathcal{U}_i \cap \mathcal{U}_j} \longrightarrow (\mathcal{F}_j)_{\mathcal{U}_i \cap \mathcal{U}_j}$$

by

$$\varphi_{ij}(f) = \left(\frac{x_i}{x_j}\right)^d f, \quad \text{for } f \in \mathbb{C}[x_0, \ldots, x_{n-1}] \text{ with } f \text{ homogeneous}.$$

<u>Theorem</u> The data $(\mathcal{S}, \mathcal{F}_i, \varphi_{ij})$ satisfy the gluing conditions and define a line bundle $\mathcal{L}_{d\omega_1}$ on $\mathbb{P}^1$.

Pic($\mathbb{P}^n$)

$$\mathrm{Pic}(\mathbb{P}^n) = \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{line bundles on } \mathbb{P}^n \end{matrix} \right\}$$

with operation given by tensor product.

Theorem The function

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathrm{Pic}(\mathbb{P}^n) \\ d &\longmapsto [\mathcal{L}_{d\omega_1}] \end{aligned}$$

is an isomorphism of abelian groups.

Global sections of $\mathcal{L}_{d\omega_1}$, $H^0(\mathbb{P}^{n-1}; \mathcal{L}_{d\omega_1})$

Let $d \in \mathbb{Z}_{\geq 0}$. The d^{th} symmetric power of $\mathbb{C}^n$ is

$$S^d(\mathbb{C}^n) = \left\{ f \in \mathbb{C}[x_0, \ldots, x_{n-1}] \;\middle|\; \begin{matrix} f \text{ is homogeneous} \\ \text{of degree } d \end{matrix} \right\}$$

$$= \left\{ f \in \mathbb{C}[x_0, \ldots, x_{n-1}] \;\middle|\; \begin{matrix} \text{if } c \in \mathbb{C}^\times \text{ then} \\ c \cdot f = c^d f \end{matrix} \right\}$$

where $(c \cdot f)(x_0, \ldots, x_n) = f(cx_0, \ldots, cx_{n-1})$.

There is a $GL_n(\mathbb{C})$-action on $H^0(\mathbb{P}^{n-1}; \mathcal{L}_{d\omega_1})$ coming from the $GL_n(\mathbb{C})$ action on lines (1-dimensional $\mathbb{C}$-submodules) of $\mathbb{C}^n$.

Theorem

(a) If $d \in \mathbb{Z}_{<0}$ then $H^0(\mathbb{P}^{n-1}; \mathcal{L}_{d\omega_1}) = 0$.

(b) If $d \in \mathbb{Z}_{\geq 0}$ then

$$H^0(\mathbb{P}^{n-1}; \mathcal{L}_{d\omega_1}) \cong S^d(\mathbb{C}^n) \text{ as } GL_n(\mathbb{C})\text{-modules}$$

Corollary

$$\dim_{\mathbb{C}}(H^0(\mathbb{P}^{n-1}; \mathcal{L}_{d\omega_1})) = \begin{cases} \binom{n-1+d}{d}, & \text{if } d \in \mathbb{Z}_{\geq 0}, \\ 0, & \text{if } d \in \mathbb{Z}_{<0}. \end{cases}$$