Algebraic Geometry Lecture 1   UniMelb.

Spaces: Philosophy

A space is a set with information about the relative position of the points.

A space is a collection of pieces glued together

A space is a ring (the ring of functions on X).

Spaces Topological spaces

A topological space $(X, \mathcal{T}_X)$ is a set X with a collection $\mathcal{T}_X$ of subsets of X such that

(a) $\phi \in \mathcal{T}_X$ and $X \in \mathcal{T}_X$

(b) If $\mathcal{S} \subseteq \mathcal{T}_X$ then $\left(\bigcup_{V \in \mathcal{S}} V\right) \in \mathcal{T}_X$

(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_\ell \in \mathcal{T}_X$ then
$$U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}_X.$$

<u>Spaces</u>: Gluing

A <u>scheme</u> is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme.

A <u>$\mathbb{K}$-variety</u> is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine $\mathbb{K}$-variety.

A <u>manifold</u> is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine manifold.

A <u>topological manifold</u> is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine topological manifold.

A <u>C^r-manifold</u> is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine C^r-manifold.

A <u>smooth manifold</u> is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine smooth manifold.

A <u>complex manifold</u> is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine complex manifold.

Spaces: Affine manifolds

The affine topological manifold is $(\mathbb{R}^n, \mathcal{T}^{std}_{\mathbb{R}^n}, C^0)$

The affine C^r-manifold is $(\mathbb{R}^n, \mathcal{T}^{std}_{\mathbb{R}^n}, C^r)$

The affine smooth manifold is $(\mathbb{R}^n, \mathcal{T}^{std}_{\mathbb{R}^n}, C^\infty)$

The affine complex manifold is $(\mathbb{C}^n, \mathcal{T}^{std}_{\mathbb{C}^n}, C^{an})$

The affine manifold is $(\mathbb{R}^n, \mathcal{T}^{std}_{\mathbb{R}^n}, C^?)$, where ? depends on who you are talking to.

<u>Spaces</u>: What does locally isomorphic mean?

Two ringed spaces $(X, \mathcal{T}_X, \mathcal{O}_X)$ and $(Y, \mathcal{T}_Y, \mathcal{O}_Y)$ are <u>locally isomorphic</u> if they satisfy:

if $p \in X$ then there exists $U \in \mathcal{T}_X$ with $p \in U$ and $V \in \mathcal{T}_Y$ and an isomorphism

$$f : (U, \mathcal{T}_U, \mathcal{O}_U) \longrightarrow (V, \mathcal{T}_V, \mathcal{O}_V)$$

- Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space. Let $V \in \mathcal{T}_X$. Define

$$\mathcal{T}_V = \{U \cap V \mid U \in \mathcal{T}_X\} \text{ and } \mathcal{O}_V(Z) = \mathcal{O}_X(Z) \text{ for } Z \in \mathcal{T}_V.$$

Then $(V, \mathcal{T}_V, \mathcal{O}_V)$ is a ringed space (see Neeman, Proposition 2.4.1)

- Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ and $(Y, \mathcal{T}_Y, \mathcal{O}_Y)$ be ringed spaces. An <u>isomorphism of ringed spaces from X to Y</u> is a homeomorphism $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and a compatible sheaf isomorphism $h : \mathcal{O}_X \to \mathcal{O}_Y$ i.e. a family of ring isomorphisms

$$h_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(f(U)) \text{ for } U \in \mathcal{T}_X$$

Spaces: What does ringed space mean?

A ringed space is a triple $(X, \mathcal{T}_X, \mathcal{O}_X)$ where $(X, \mathcal{T}_X)$ is a topological space and $\mathcal{O}_X \in Sh(X)$.

The structure sheaf of $(X, \mathcal{T}_X, \mathcal{O}_X)$ is $\mathcal{O}_X$.

Presheaves and Sheaves

Let $(X, \mathcal{T}_X)$ be a topological space.

$\mathcal{T}_X$ is a category with morphisms inclusions,

Objects: $U \in \mathcal{T}_X$

Morphisms: $Hom(U,V) = \emptyset$, if $U \not\subseteq V$

$Hom(U,V) = \{\iota_{UV}\}$, if $U \subseteq V$,

where $\iota_U^V : U \to V$ is the inclusion.

Let $\mathcal{A}$ be the category of commutative rings with 1.

$PreSh(X) = \{$contravariant functors $\mathcal{F} : \mathcal{T}_X \to \mathcal{A}\}$

$Sh(X) = \{$exact contravariant functors $\mathcal{F} : \mathcal{T}_X \to \mathcal{A}\}$.

Affine Schemes

An *affine scheme* is an element of

$$\mathrm{im}(\mathrm{Spec}).$$

Spec is the contravariant functor

$$\begin{aligned}\mathrm{Spec}: \left\{\begin{matrix}\text{commutative}\\ \text{rings}\end{matrix}\right\} &\longrightarrow \{\text{ringed spaces}\}\\ A &\longmapsto (X, \mathcal{T}_X, \mathcal{O}_X)\\ \varphi: A_1 \to A_2 &\longmapsto \mathrm{Spec}(A_2) \to \mathrm{Spec}(A_1)\\ & \qquad\quad p \longmapsto \varphi^{-1}(p)\end{aligned}$$

given by

$$X = \mathrm{Spec}(A) = \{\text{prime ideals of } A\}$$

$\mathcal{T}_X^{\mathrm{Zar}}$ has closed sets

$$V(S) = \{p \in X \mid x = 0 \text{ in } \tfrac{A}{p}\} \quad \text{for } S \subseteq A$$

$\mathcal{O}_X$ is determined by

$$\mathcal{O}_X(X_q) = A[\tfrac{1}{q}] \quad \text{and} \quad \mathrm{res}^{X_q}_{X_k}: A[\tfrac{1}{k}] \longrightarrow A[\tfrac{1}{q}]$$

$$\frac{f}{k^m} \longmapsto \frac{f s^m}{q^{mn}}$$

if $q^n = sk$ with $s \in A$ and $n \in \mathbb{Z}_{>0}$.

Let A be a commutative ring. Let $g \in A$.

The <u>basic set</u> is

$$X_g = \{ \mathfrak{p} \in X \mid g \neq 0 \text{ in } \tfrac{A}{\mathfrak{p}} \}$$

$$= \{ \mathfrak{p} \in X \mid g \notin \mathfrak{p} \}$$

The <u>basic ring</u> is

$$A[\tfrac{1}{g}] = \{ \tfrac{f}{g^k} \mid f \in A,\ k \in \mathbb{Z}_{\geq 0} \}$$

with

$$\frac{f_1}{g^k} = \frac{f_2}{g^l} \quad \text{if there exists } n \in \mathbb{Z}_{>0} \text{ such that } f_1 g^{l+n} = f_2 g^{k+n}$$

$$\frac{f_1}{g^k} + \frac{f_2}{g^l} = \frac{f_1 g^l + f_2 g^k}{g^{k+l}} \quad \text{and} \quad \frac{f_1}{g^k} \cdot \frac{f_2}{g^l} = \frac{f_1 f_2}{g^{k+l}}.$$

with the ring homomorphism

$$\begin{aligned} \iota : A &\longrightarrow A[\tfrac{1}{g}] \\ f &\longmapsto \frac{f}{1}. \end{aligned}$$

A <u>contravariant functor</u> $\mathcal{F}: \mathcal{T}_X \to \mathcal{A}$ is a

function
$$\mathcal{F}: \mathcal{T}_X \to \mathcal{A}$$
$$U \mapsto \mathcal{F}(U)$$

and a collection of functions
$$\mathcal{F}: \mathrm{Hom}(U,V) \longrightarrow \mathrm{Hom}(\mathcal{F}(U), \mathcal{F}(V))$$
$$\iota_U^V \longmapsto \mathrm{res}_U^V$$

such that
$$\mathcal{F}(id_U) = \mathcal{F}(\iota_U^U) = \mathrm{res}_U^U = id_{\mathcal{F}(U)} \quad \text{and}$$
$$\mathcal{F}(\iota_V^W \iota_U^V) = \mathcal{F}(\iota_U^V) \circ \mathcal{F}(\iota_V^W) = \mathrm{res}_U^V \circ \mathrm{res}_V^W$$
$$= \mathcal{F}(\iota_U^W) = \mathrm{res}_U^W$$

An <u>exact contravariant functor</u> $\mathcal{F}$ is a contravariant functor $\mathcal{F}: \mathcal{T}_X \to \mathcal{A}$ such that

if $U \in \mathcal{T}_X$ and $\mathcal{S}$ is an open cover of U then
$$\mathrm{im}(\rho_0^{U,\mathcal{S}}) = \ker(\rho_1^{U,\mathcal{S}}, \rho_2^{U,\mathcal{S}})$$

where
$$\ker(\rho_1^{U,\mathcal{S}}, \rho_2^{U,\mathcal{S}}) = \{ (f_{V\cap U})_{V\in\mathcal{S}} \mid \rho_1^{U,\mathcal{S}}((f_{V\cap U})_{V\in\mathcal{S}}) = \rho_2^{U,\mathcal{S}}((f_{V\cap U})_{V\in\mathcal{S}}) \}.$$

where

$$\mathcal{F}(\mathcal{U}) \xrightarrow{p_0^{\mathcal{U},\mathcal{S}}} \prod_{V\in\mathcal{S}} \mathcal{F}(V\cap\mathcal{U}) \underset{p_2^{\mathcal{U},\mathcal{S}}}{\overset{p_1^{\mathcal{U},\mathcal{S}}}{\rightrightarrows}} \prod_{W,Z\in\mathcal{S}} \mathcal{F}(W\cap Z\cap\mathcal{U})$$

$$f \overset{p_0^{\mathcal{U},\mathcal{S}}}{\longmapsto} \left(\mathrm{res}^{V}_{V\cap\mathcal{U}}(f)\right)_{V\in\mathcal{S}}$$

$$(f_{V\cap\mathcal{U}})_{V\in\mathcal{S}} \overset{p_1^{\mathcal{U},\mathcal{S}}}{\longmapsto} \left(\mathrm{res}^{V\cap\mathcal{U}}_{V\cap Z\cap\mathcal{U}}(f_{V\cap\mathcal{U}})\right)_{V,Z\in\mathcal{S}}$$

$$(f_{V\cap\mathcal{U}})_{V\in\mathcal{S}} \underset{p_2^{\mathcal{U},\mathcal{S}}}{\longmapsto} \left(\mathrm{res}^{V\cap\mathcal{U}}_{W\cap V\cap\mathcal{U}}(f_{V\cap\mathcal{U}})\right)_{W,V\in\mathcal{S}}.$$

Coherent sheaves: Chapter 7 of Neeman

A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is a sheaf

(exact contravariant functor $\mathcal{F}: \mathcal{T}_X \to \left\{ \begin{matrix} \mathbb{Z}\text{-linear} \\ \text{spaces} \end{matrix} \right\}$)

such that

(a) If $U \in \mathcal{T}_X$ then $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module,

(b) the morphisms

$$res_U^V : \mathcal{F}(V) \to \mathcal{F}(U) \quad \text{for } U \subseteq V$$

satisfy

$$res_U^V(fm) = res_U^V(f)\, res_U^V(m)$$

for $f \in \mathcal{O}_X(V)$ and $m \in \mathcal{F}(V)$.

A locally free sheaf on X, or

vector bundle on X, is a sheaf of ~~$\mathcal{O}_X$-mod~~ $\mathcal{O}_X$-modules $\mathcal{F}$ such that

if $p \in X$ then there exists $U \in \mathcal{T}_X$ with $p \in U$ such that

$$\mathcal{F}(U) \text{ is a free } \mathcal{O}_X(U)\text{-module.}$$

A <u>coherent sheaf on X</u> is

a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that

if $p \in X$ then there exists $U \in \mathcal{T}_X$ with $p \in U$

such that

$\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module.

The condition $\mathcal{F}(U)$ is a <u>free</u> $\mathcal{O}_X(U)$-module

means that

there exists $n \in \mathbb{Z}_{>0}$ and $e_1, \ldots, e_n \in \mathcal{F}(U)$

such that

(a) $\mathcal{F}(U) = \mathcal{O}_X(U)\text{-span}\{e_1, \ldots, e_n\}$

(b) $e_1, \ldots, e_n$ are $\mathcal{O}_X(U)$-linearly independent.

The condition $\mathcal{F}(U)$ is a <u>finitely generated</u>

$\mathcal{O}_X(U)$-module means that

there exists $n \in \mathbb{Z}_{>0}$ and $e_1, \ldots, e_n \in \mathcal{F}(U)$

such that

$$\mathcal{F}(U) = \mathcal{O}_X(U)\text{-span}\{e_1, \ldots, e_n\}.$$