Algebraic Geometry Week 12

17.10.2018
Uni Melb
A. Ram

Projective space $\mathbb{P}^1$

$\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by $c(z_1,z_2) = (cz_1, cz_2)$. Let

$$\mathbb{P}^1 = \frac{\mathbb{C}^2 - \{(0,0)\}}{\mathbb{C}^\times} \quad \text{and} \quad \begin{array}{ccc} \mathbb{C}^2-\{(0,0)\} & \xrightarrow{\pi} & \mathbb{P}^1 \\ (a,b) & \longmapsto & [a,b] = \mathbb{C}^\times(a,b). \end{array}$$

Give $\mathbb{P}^1$ the quotient topology from $\mathbb{C}^2 - \{0,0\}$,

$$\mathcal{T}_{\mathbb{P}^1} = \{ U \subseteq \mathbb{P}^1 \mid \pi^{-1}(U) \in \mathcal{T}_{\mathbb{C}^2-\{(0,0)\}} \}$$

Let

$$U_0 = \{ [z,1] \in \mathbb{P}^1 \mid z \in \mathbb{C} \} \simeq \mathbb{C}$$
$$U_\infty = \{ [1,z] \in \mathbb{P}^1 \mid z \in \mathbb{C} \} \simeq \mathbb{C}, \quad \text{and}$$

$$U_0 \cap U_\infty = \{ [z,1] = [1,z^{-1}] \in \mathbb{P}^1 \mid z \in \mathbb{C}^\times \} \simeq \mathbb{C}^\times.$$

Algebraic version Make U_0 into a ringed space by

$(U_0, \mathcal{T}_{U_0}, \mathcal{O}_{U_0}) = \mathrm{Spec}(\mathbb{C}[z])$ and

$(U_\infty, \mathcal{T}_{U_\infty}, \mathcal{O}_{U_\infty}) = \mathrm{Spec}(\mathbb{C}[z])$ and

glue these together over

$$(U_0 \cap U_\infty, \mathcal{T}_{U_0\cap U_\infty}, \mathcal{O}_{U_0\cap U_\infty}) = \mathrm{Spec}(\mathbb{C}[z,z^{-1}])$$

via the maps

$$\mathbb{C}[z] \hookrightarrow \mathbb{C}[z,z^{-1}] \hookleftarrow \mathbb{C}[z^{-1}].$$

Algebraic Geometry Week 12   17.10.2018 UniMelb A. Ram   2

Projective space: Analytic version

Define $\mathcal{O}_{\mathbb{P}^1}^{an}$ by

$$\mathcal{O}_{\mathbb{P}^1}^{an}(U) = \left\{ f: U \to \mathbb{C} \;\middle|\; \begin{array}{l} f: U_0 \cap U \to \mathbb{C} \text{ and} \\ f: U_\infty \cap U \to \mathbb{C} \text{ are} \\ \text{holomorphic} \end{array} \right\}$$

$$= \{ f: U \to \mathbb{C} \mid \pi^{-1}(U) \xrightarrow{f\circ\pi} \mathbb{C} \text{ is holomorphic} \}.$$

Then, by Neeman Remark 5.8.21,

$$\mathcal{O}_{\mathbb{P}}^{an}(U_0) = \left\{ f = \sum_{n\in\mathbb{Z}_{\ge 0}} a_n z^n \;\middle|\; \begin{array}{l} \text{if } B \in \mathbb{R}_{\ge 1} \text{ then there exists} \\ M \in \mathbb{R}_{>0} \text{ such that} \\ \text{if } n \in \mathbb{Z}_{\ge 0} \text{ then } |a_n| \le \frac{M}{B^n} \end{array} \right\}$$

$$\mathcal{O}_{\mathbb{P}}^{an}(U_\infty) = \left\{ f = \sum_{n\in\mathbb{Z}_{\le 0}} a_n z^n \;\middle|\; \begin{array}{l} \text{if } B \in \mathbb{R}_{\ge 1} \text{ then there exists} \\ M \in \mathbb{R}_{>0} \text{ such that} \\ \text{if } n \in \mathbb{Z}_{\le 0} \text{ then } |a_n| \le \frac{M}{B^{|n|}} \end{array} \right\}$$

$$\mathcal{O}_{\mathbb{P}}^{an}(U_0 \cap U_\infty) = \left\{ f = \sum_{n\in\mathbb{Z}} a_n z^n \;\middle|\; \begin{array}{l} \text{if } B \in \mathbb{R}_{\ge 1} \text{ then there exists} \\ M \in \mathbb{R}_{>0} \text{ such that} \\ \text{if } n \in \mathbb{Z} \text{ then } |a_n| \le \frac{M}{B^{|n|}} \end{array} \right\}$$

and the complex manifold structure is given by

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^1}^{an}(U_0) & \hookrightarrow & \mathcal{O}_{\mathbb{P}^1}^{an}(U_0 \cap U_\infty) & \hookleftarrow & \mathcal{O}_{\mathbb{P}^1}^{an}(U_\infty) \\ f(z) & \longmapsto & f(z) & & \\ & & g(z^{-1}) & \longleftarrow & g(z^{-1}) \end{array}$$

GAGA (Analytic geometry Algebraic Geometry)

Let $\mathcal{T}_{\mathbb{C}^n}$ be a topology on $\mathbb{C}^n$ such that $\mathcal{T}_{\mathbb{C}^n} \supseteq \mathcal{T}^{Zar}_{\mathbb{C}^n}$

Let $U \in \mathcal{T}_{\mathbb{C}^n}$.

A holomorphic / polynomial function on U is

a function $f: U \to \mathbb{C}$ such that if $p \in U$ then

there exists $V \in \mathcal{T}_{\mathbb{C}^n}$ with $V \subseteq U$ and $p \in V$ and

$a_{i_1 \cdots i_n} \in \mathbb{C}$ for $i_1, \ldots, i_n \in \mathbb{Z}_{\geq 0}$

such that

all but a finite number of the $a_{i_1 \cdots i_n}$ are zero and

if $z \in V$ then $f(z) = \sum_{i_1, \ldots, i_n \in \mathbb{Z}} a_{i_1 \cdots i_n} (z_1 - p_1)^{i_1} \cdots (z_n - p_n)^{i_n}$.

Neeman's definition p.4 of meromorphic function seems to say:

A meromorphic / regular function on U is

a function $h: U \to \mathbb{C}$ such that if $p \in U$ then

there exists $V \in \mathcal{T}_{\mathbb{C}^n}$ with $V \subseteq U$ and $p \in V$ and

f and g holomorphic / polynomial functions on V

such that

if $z \in V$ then $g(z) \neq 0$ and $h(z) = \dfrac{f(z)}{g(z)}$

The sheaf $\mathcal{O}^{hol}_{\mathbb{C}^n}$ on $(\mathbb{C}^n, \mathcal{T}^{std}_{\mathbb{C}^n})$

For $U \in \mathcal{T}^{std}_{\mathbb{C}^n}$ define

$$\mathcal{O}^{hol}_{\mathbb{C}^n}(U) = \{ f : U \to \mathbb{C} \mid f \text{ is holomorphic on } U \}$$

with
$$\begin{aligned} res^V_U : \mathcal{O}^{hol}_{\mathbb{C}^n}(V) &\longrightarrow \mathcal{O}^{hol}_{\mathbb{C}^n}(U) \\ f &\longmapsto f : U \to \mathbb{C} \\ & \quad\ u \mapsto f(u) \end{aligned} \qquad \text{for } U \subseteq V.$$

Then $(\mathbb{C}^n, \mathcal{T}^{std}_{\mathbb{C}^n}, \mathcal{O}^{hol}_{\mathbb{C}^n})$ is a ringed space.

<u>HW</u> Show that $\mathbb{C}^n = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n\text{-times}}$ as ringed spaces. (see Neeman Caution 8.1.4).
This is <u>not</u> true if the ringed space structure on $\mathbb{C}^n$ is $(\mathbb{C}^n, \mathcal{T}^{Zar}_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n}) = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n])$

Alg. geom. Week 12 17.10.2018

UniMelb A. Ram (4)

Cartan's definition Ch VI §4.5 of meromorphic function is as follows:

Let $(X, \mathcal{T}_X, \mathcal{O}_X^{an})$ be a complex manifold.

A meromorphic function on X is a morphism of complex manifolds

$$f : X \longrightarrow \mathbb{P}^1.$$

Harder Chapt 7 p. 64 has a definition of meromorphic functions as follows:

Let X be an affine scheme of finite type

$$X = \mathrm{Spec}\left(\frac{\mathbb{F}[x_1, \ldots, x_n]}{I}\right) = \mathrm{Spec}(A)$$

(or an irreducible reduced scheme of finite type).

If $U \in \mathcal{T}_X$ is an affine subscheme and $U \neq \emptyset$ then

$$\mathrm{Frac}(\mathcal{O}_X(U)) \text{ is independent of } U.$$

The field of meromorphic functions on X, or function field of X is

$$\mathrm{Frac}(\mathcal{O}_X(U)).$$

Harder's reference to meromorphic functions

Harder p. 199 Volume I lets

$$\mathbb{C}(\mathbb{P}^1) = \{\text{meromorphic functions on } \mathbb{P}^1\}$$

and states

$$\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(z) = \left\{\frac{f}{g} \mid f, g \in \mathbb{C}[z]\right\}$$

Then Harder says: For a Riemann surface S and $V \in \mathcal{T}_S$ let $\mathbb{C}(S) = \{\text{meromorphic functions on } S\}$

$$\mathcal{O}_S^{mer}(V) = \{f \in \mathbb{C}(S) \mid f: V \to \mathbb{C} \text{ is holomorphic}\}$$

Then he gives

$$\mathcal{O}_{\mathbb{P}^1}^{mer}(U_0) = \mathbb{C}[z] \qquad \mathcal{O}_{\mathbb{P}}^{mer}(U_\infty) = \mathbb{C}[z^{-1}]$$

$$\mathcal{O}_{\mathbb{P}^1}^{mer}(U_0 \cap U_\infty) = \mathbb{C}[z, z^{-1}]$$

and $\mathcal{O}_{\mathbb{P}^1}^{mer}(\mathbb{P}^1) = \mathbb{C}$.

Harder says, for a compact Riemann surface S

$K = \mathbb{C}(S)$ is a field

$$\mathrm{Val}(K) = \left\{\text{subrings } A \subseteq K \;\middle|\; \begin{array}{l}\text{if } f \in K \text{ then } f \in A \text{ or } f^{-1} \in A, \\ A \neq K \text{ and } A \supseteq \mathbb{C}\end{array}\right\}$$

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & \mathrm{Val}(K) \\ p & \longmapsto & \mathcal{O}_p^{mer} \end{array} \quad \text{where}$$

$$\mathcal{O}_p^{mer} = \{f \in \mathbb{C}(S) \mid f \text{ is regular at } p\}.$$

Alg. Geom. Week 12 17.10.2018 Uni Melb A. Ram (16)

Define

$$\mathcal{T}^{zar}_{Val(K)} = \{ \mathcal{U} \subseteq Val(K) \mid \mathcal{U}^c \text{ is finite} \}.$$

$$\mathcal{O}^{mer}_S(\mathcal{U}) = \bigcap_{A \in \mathcal{U}} A \qquad \begin{pmatrix}\text{functions regular on } \mathcal{U} \\ \text{and meromorphic on } S\end{pmatrix}$$

There is a given map of ringed spaces

$$(S, \mathcal{T}^{std}_S, \mathcal{O}^{hol}_S) \longrightarrow (Val(K), \mathcal{T}^{zar}_{Val(K)}, \mathcal{O}^{mer}_S)$$

which is a bijection $S \leftrightarrow Val(K)$

Note that

$\mathcal{T}^{std}_S$ is the topology generated by

$\{ \mathcal{T}_{\mathcal{U}} \mid \mathcal{U} \in \mathcal{T}^{zar}_{Val(K)} \}$ where $\mathcal{T}_{\mathcal{U}}$ is the coarsest topology such that all elements of $\mathcal{O}^{mer}_S(\mathcal{U})$ are continuous.

Alg. Geom. Week 12 17.10.2018 UniMelb A. Ram (7)

Vector bundles and local systems

Let C_n^* be the sheaf of continuous functions $X \to Gl_n(\mathbb{C})$ where $Gl_n(\mathbb{C})$ has the topology coming from $\mathbb{C}$ via $Gl_n(\mathbb{C}) \subseteq \mathbb{C}^{n^2}$; discrete topology (locally constant functions)

Let $\mathcal{S}$ be an open cover of X. An <u>$\mathcal{S}$ 1-cocycle</u> is a collection of continuous maps

$$g_{UV}: U \cap V \longrightarrow Gl_n(\mathbb{C}) \quad \text{for } U, V \in \mathcal{S}$$

such that

(a) if $U \in \mathcal{S}$ and $p \in U$ then $g_{UU}(p) = 1$,

(b) if $U, V, W \in \mathcal{S}$ and $p \in U \cap V \cap W$ then

$$g_{UV}(p)\, g_{VW}(p) = g_{UW}(p).$$

A <u>vector bundle</u> / <u>local system</u> of rank n on X is an $\mathcal{S}$ 1-cocycle for an open cover $\mathcal{S}$.

If $g = (g_{UV})$ is a vector bundle $g_{UV} \in Gl_n(\mathcal{O}_{U\cap V})$

If $g = (g_{UV})$ is a local system $g_{UV} \in Gl_n(\mathbb{C})$

Alg. Geom. Week 12 12.10.2018 UniMelb A. Ram

Connections

Let E be a vector bundle on X.

A <u>connection on E</u> is a $\mathbb{C}$-module homomorphism

$$\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^1_X$$

such that

$$\nabla(mf) = \nabla(m)f + m\,df, \quad \text{for } f \in \mathcal{O}_X,\ m \in E.$$

Define

$$0 \to E \xrightarrow{\nabla} E \otimes \Omega^1_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} E \otimes \Omega^{top}_X \to 0$$

by

$$\nabla(m \otimes \omega) = m \otimes d\omega + \nabla(m) \wedge \omega$$

for $m \in E$ and $\omega \in \Omega^p_X$. The connection ∇ is <u>flat</u> if $\nabla^2 = 0$.

In coordinates the connection can be given by

$$\nabla(e_i) = e_1 \otimes A_{1i} + \cdots + e_n \otimes A_{ni} \quad \text{with } A_{ij} \in \Omega^1_X$$

and

$$\nabla(m) = (d - A)m.$$

A <u>parallel section</u> is $m \in E(U)$ such that

$$\nabla(m) = 0, \quad \text{i.e.} \quad dm = Am$$

so that $\frac{1}{m} dm = A$ and $\log(m) = \int A$

and $m = e^{\int A}$.

Let E be a vector bundle. As a sheaf, locally

$$E(U) \cong \mathcal{O}_U^{\oplus n}.$$

Let K be a local system. As a sheaf, locally

$$K(U) \cong \mathbb{C}^{\oplus n}.$$

and $E = K \otimes_{\mathbb{C}} \mathcal{O}_X$ is a vector bundle.

<u>Riemann-Hilbert correspondence</u>

$$\left\{\begin{array}{l}(E,\nabla) \text{ with} \\ E \text{ a vector bundle} \\ \nabla: E \to E\otimes\Omega_X^1 \text{ a } \underline{\text{flat}} \\ \text{connection}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{local systems} \\ (\text{of } \mathbb{C}\text{ vectorspaces})\end{array}\right\}$$

$$(E,\nabla) \longmapsto K = \ker \nabla.$$

$$(K\otimes_{\mathbb{C}}\mathcal{O}_X, 1\otimes d) \longleftarrow K$$

Here $K(U) = \{s \in E(U) \mid \nabla(s) = 0\}$ $\left(\begin{array}{l}\text{parallel} \\ \text{sections}\end{array}\right)$

If $e_1, \ldots, e_n$ is a local basis of $K(U)$ then $\nabla = 1\otimes d$ on $E = K\otimes_{\mathbb{C}}\mathcal{O}_X$ is given by

$$\nabla(e_1f_1 + \cdots + e_nf_n) = e_1\otimes df_1 + \cdots + e_n\otimes df_n.$$

The Riemann-Hilbert correspondence generalises to

$$\left\{\begin{matrix}\text{holonomic}\\ D\text{-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{perverse}\\ \text{sheaves}\end{matrix}\right\}$$

$$\cup\!\!| \qquad\qquad\qquad \cup\!\!|$$

$$\left\{\begin{matrix}(E,\nabla)\text{ vector bundle}\\ \text{with flat connection}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{local}\\ \text{systems}\end{matrix}\right\}$$

<u>Local systems $\leftrightarrow$ representations of $\pi_1(X)$</u> (see Harder §4.8.1 p.127)

Let $x_0 \in X$ (a "base point"). The <u>fundamental group of X</u> with base point x_0 is

$$\pi_1(X, x_0) = \frac{\left\{ \gamma : [0,1] \to X \;\middle|\; \begin{matrix}\gamma(0)=\gamma(1)=x_0\\ \gamma \text{ is continuous}\end{matrix} \right\}}{\text{homotopy}}$$

with product

$$(\gamma_2\gamma_1)(t) = \begin{cases} \gamma_1(2t), & \text{if } t \in \mathbb{R}_{[0,\frac{1}{2}]} \\ \gamma_2(2(t-\frac{1}{2})), & \text{if } t \in \mathbb{R}_{[\frac{1}{2},1]} \end{cases}$$

More generally, let $x, y \in X$. A <u>path from x to y in X</u> is a continuous function

$$\gamma : \mathbb{R}_{[0,1]} \to X \quad \text{with} \quad \gamma(0) = x \text{ and } \gamma(1) = y.$$

Local systems to representations of $\pi_1(X, x_0)$

Let K be a local system in X, given by an open cover $\mathcal{S}$ and $g = (g_{UV})_{U,V\in\mathcal{S}}$.

Let $\gamma : \mathbb{R}_{[0,1]} \to X$ be a path in X.

Let K_x be the stalk of K at x,
K_y the stalk of K at y ($K_x \cong \mathbb{C}^n$, $K_y \cong \mathbb{C}^n$)

The local system K gives an isomorphism
$$\Psi_\gamma : K_x \to K_y \quad \text{given by}$$

$$\Psi_\gamma = g_{V_\ell V_{\ell-1}} \cdots g_{V_3V_2}\, g_{V_2V_1}$$

This depends only on the homotopy ~~typ~~ class of the path. Then

$$\rho : \pi_1(X, x_0) \longrightarrow \mathrm{Aut}(K_{x_0}) = \mathrm{Aut}(\mathbb{C}^n) = GL_n(\mathbb{C})$$
$$\gamma \longmapsto \Psi_\gamma$$

is a representation of $\pi_1(X, x_0)$.

This is the <u>monodromy</u> representation of K.

Representations of $\pi_1(X, x_0)$ to local systems

Let $\rho: \pi_1(X, x_0) \to GL_n(\mathbb{C})$

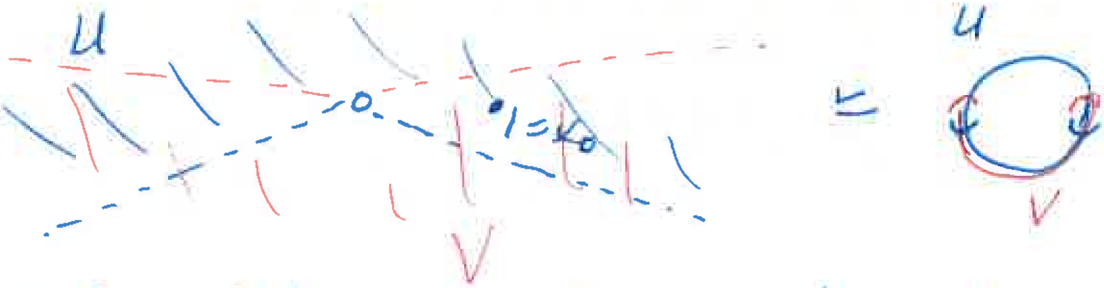
be a representation (group homomorphism).

Define a local system K on X by setting the stalk K_x of K at x to be

$$K_x = \frac{\{(\gamma, v) \mid v \in \mathbb{C}^n \text{ and } \gamma: \mathbb{R}_{[0,1]} \to X \text{ with } \gamma(0) = x_0, \gamma(1) = x\}}{\{[\gamma, v_1] = [\sigma, v_2] \text{ if } \rho(\sigma^{-1} \circ \gamma) v_1 = v_2\}}$$

Favourite example

$X = \mathbb{C}^\times$, $x_0 = 1$, so that $\pi_1(X, x_0) = \mathbb{Z}$.



$\mathcal{S} = \{U, V\}$ and there is only one intersection $U \cap V$

$$\begin{aligned} g_{UV}: U \cap V &\to (GL_n(\mathbb{C}), \mathcal{T}^{disc}_{GL_n}) \\ C_a &\mapsto g_{UV}^{(a)} \\ C_b &\mapsto g_{UV}^{(b)} \end{aligned}$$

where C_a and C_b are the connected components of $U \cap V$.

By a change of trivialisation $h = (h_U, h_V)$ we can assume $g_{UV}^{(a)} = 1$. So a local system on X is a single matrix $g_{UV}^{(b)} \in GL_n(\mathbb{C})$.

Let $a = \log(g_{uv}^{(b)})$. The corresponding flat connection $\nabla : E \to E \otimes \Omega_X^1$ is given by

$\nabla = d - a\,dz$ so that $\nabla(s) = ds - s\,a\,dz$

since $\Omega_X^1 = \mathcal{O}_X\text{-span}\{dz\}$ and

$s = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ and $a = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ and the equation

$\nabla(s) = 0$ is $\begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} = \begin{pmatrix} a_{11}dz & \cdots & a_{1n}dz \\ & \vdots & \\ a_{n1}dz & \cdots & a_{nn}dz \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

which is $\begin{pmatrix} \frac{df_1}{dz} \\ \vdots \\ \frac{df_n}{dz} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

which has solution $\vec{f} = e^{az}$ I think.
(this is easy to check if $n=1$).

<u>Two examples</u> ~~are~~ on $X = \mathbb{C}$ are

$\nabla = d$ so $\nabla(f) = 0$ is $\frac{df}{dz} = 0$ so f is constant.

and

$\nabla = d - 1$ so $\nabla(f) = 0$ is $\frac{df}{dz} = f$. So $f = ce^z \in \mathbb{C}e^z$.

This are analytically isomorphic local systems on $\mathbb{C}$ but not algebraically isomorphic).