Alg. geom. week 11 10.10.2018 UniMelb A. Ram ①

G-modules, weight vectors and highest weight vectors

Let G be a group.
A <u>G-module</u> is a vector space V with an action of G on V by linear transformations, i.e.

$$\begin{array}{rcl} G\times V & \to & V \\ (g,v) & \mapsto & gv \end{array} \quad \text{with } g(c_1v_1+c_2v_2) = c_1(gv_1)+c_2(gv_2)$$

for $c_1,c_2\in\mathbb{C}$ and $v_1,v_2\in V$.

Let $G = GL_n(\mathbb{C})$,

$$B = \left\{\begin{pmatrix} x_1 & & u_{ij} \\ & \ddots & \\ 0 & & x_n\end{pmatrix}\in GL_n(\mathbb{C})\right\},\quad T = \left\{\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n\end{pmatrix}\in GL_n(\mathbb{C})\right\}$$

~~Let~~ For $\lambda = (\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n$ let
$X^\lambda : B\to\mathbb{C}^\times$ be given by $X^\lambda\begin{pmatrix} x_1 & & u_{ij} \\ & \ddots & \\ 0 & & x_n\end{pmatrix} = x_1^{\lambda_1}\cdots x_n^{\lambda_n}$

Let V be a G-module. A <u>weight vector of weight λ</u> is a vector $v\in V$ such that

if $h\in T$ then $hv = X^\lambda(h)v$,

i.e. an irreducible T-submodule of V.

A <u>highest weight vector of weight λ</u> is a vector $v\in V$ such that

if $b\in B$ then $bv = X^\lambda(b)v$,

i.e. an irreducible B-submodule of V

Hermann Weyl's theorem (For this $G = GL_n(\mathbb{C})$ case this is earlier than Hermann Weyl).

(a) Let V be an irreducible finite dimensional G-module. Then V contains a unique (up to scalar multiples) highest weight vector.

(b) There exists an irreducible G-module $L(\lambda)$ with highest weight vector of weight λ if and only if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Favorite construction of vector bundles on G/B

Let V be a B-module. Define a vector bundle on G/B by

$$\begin{array}{cc} G\times_B V & [g,v] \\ \downarrow & \downarrow \\ G/B & gB \end{array} \quad \text{where } G\times_B V = \frac{G\times V}{\left\langle [gb,v]=[g,bv] \text{ for } \begin{array}{l} g\in G \\ b\in B \\ v\in V\end{array} \right\rangle}$$

A global section of $G\times_B V$ is

$$\tilde{s}: G/B \to G\times_B V \quad \text{such that} \quad \pi\circ\tilde{s} = id_{G/B}.$$

Let $\tilde{s}$ be a global section of $G\times_B V$ and define

$$s: G\to V \quad \text{by} \quad \tilde{s}(gB) = [g, s(g)].$$

If $b\in B$ then

$$[g,s(g)] = \tilde{s}(gB) = \tilde{s}(gbB) = [gb, s(gb)]$$
$$= [g, bs(gb)], \quad \text{so that}$$

s satisfies $\quad s(gb) = b^{-1}s(g)$.

In this way

$$H^0(G/B, V) \longleftrightarrow \left\{ \begin{array}{l} \text{functions } s: G\to V \text{ such} \\ \text{that } s(gb) = b^{-1}s(g) \end{array} \right\}.$$

G acts on functions $s: G\to V$ by

$$(gs)(v) = s(g^{-1}v), \quad \text{for } g\in G, v\in V$$

and so $H^0(G/B, V)$ is a G-module.

The elements s_γ in $H^0(G/B, \mathcal{L}_\lambda)$

Let $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$, where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ and

$$\mathbb{C}_\lambda = \mathbb{C}\text{-span}\{v_\lambda\} \text{ with } b v_\lambda = X^\lambda(b) v_\lambda$$

is a 1-dimensional (irreducible) B-module.

The set

$$U_{w_0} = B w_0 B = \left\{ u^+ w_0 B \;\middle|\; u^+ = \begin{pmatrix} 1 & \cdot & u_{ij} \\ 0 & \cdot & 1 \end{pmatrix}, u_{ij} \in \mathbb{C} \right\} \cong \mathbb{C}^{\binom{n}{2}}$$

is a dense open set of G/B.

A polynomial function on U_{w_0} is an element of $\mathbb{C}[u_{12}, u_{13}, \ldots, u_{n-1,n}]$.

Fix positive integers $\gamma_{ij} \in \mathbb{Z}_{\geq 0}$ for $i, j \in \{1, \ldots, n\}$ with $i<j$. Define

$$s_\gamma : G \to \mathbb{C} \quad \text{by} \quad s_\gamma\left(\begin{pmatrix} 1 & \cdot & u_{ij} \\ & \cdot & 1 \end{pmatrix} w_0 \right) = u_{12}^{\gamma_{12}} u_{13}^{\gamma_{13}} \cdots u_{n-1,n}^{\gamma_{n-1,n}}.$$

and $s_\gamma(gb) = s_\gamma(g) X^\lambda(b^{-1})$ for $g \in G$ and $b \in B$ and continuity.

The group G acts on the vector space

$$L(\lambda) = \mathbb{C}\text{-span}\{s_\gamma\} \quad \text{by} \quad (g s_\gamma)(k) = s_\gamma(g^{-1}k)$$

for $k, g \in G$.

Proposition Let $\gamma = (\gamma_1, \ldots, \gamma_n) = \sum_{1\le i<j\le n} \gamma_{ij}(\varepsilon_i - \varepsilon_j)$

(a) s_γ is a weight vector of weight $-w_0\lambda + \gamma$

(b) s_γ is a highest weight vector if and only if $\gamma_{ij} = 0$ for $i,j \in \{1,\ldots,n\}$ with $i<j$.

Proof of (b) Assume s_γ is a highest weight vector. Let $u \in U^+$. Then

$$s_\gamma(uw_0) = (u^{-1}s_\gamma)(w) = s_\gamma(w_0).$$

So s_γ is a constant function on Uw_0. So $\gamma_{ij} = 0$ and $s_\gamma = 1$.

Proof of (a) Let $h = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ and $u = \begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}$. Then

$$(hs_\gamma)(uw_0) = s_\gamma(h^{-1}uw_0) = s_\gamma((h^{-1}uh)w_0 w_0 h^{-1} w_0)$$

$$= s_\gamma((h^{-1}uh)w_0)\, X^\lambda(w_0 h^{-1} w_0)$$

$$= \Big(\prod_{1\le i<j\le n} (u_{ij} x_i^{-1} x_j)^{\gamma_{ij}}\Big) x^{-w_0\lambda} = s_\gamma(u) x^{-w_0\lambda + \gamma}$$

Note that all weights of $L(\lambda)$ are in

$$-w_0\lambda + \sum_{1\le i\le j\le n} \mathbb{Z}_{\ge 0}(\varepsilon_i - \varepsilon_j) = -w_0\lambda - Q^+$$

where $Q^+ = \mathbb{Z}_{\ge 0}\text{span}\{\varepsilon_i - \varepsilon_j \mid 1 \le i<j\le n\}$.

If $L(\lambda)$ is a G-module and $w \in W$
then $ws_0 \in L(\lambda)$.

Since $(h(ws_0)) = ws_0 \cdot X^{-ww_0\lambda}$, ws_0 is a weight vector
then $-ww_0\lambda$ must be in $-w_0\lambda - Q^+$.

So $w_0\lambda - ww_0\lambda \in -Q^+$

So $ww_0\lambda - w_0\lambda \in Q^+$.

If $\lambda = (\lambda_1, \ldots, \lambda_n)$ then $w_0\lambda = (\lambda_n, \ldots, \lambda_1)$.

If $w = s_{n-i}$ then

$$
\begin{aligned}
ww_0\lambda - w_0\lambda &= s_{n-i}w_0\lambda - w_0\lambda \\
&= (\lambda_n, \ldots, \lambda_{i+1}, \lambda_i, \ldots \lambda_1) - (\lambda_n, \ldots, \lambda_1) \\
&= (0, \ldots, 0, \lambda_{i+1} - \lambda_i, \lambda_i - \lambda_{i+1}, 0, \ldots 0) \\
&= (\lambda_{i+1} - \lambda_i)(\varepsilon_{n-i} - \varepsilon_{n-i+1})
\end{aligned}
$$

and this is in Q^+ exactly when $\lambda_{i+1} - \lambda_i \geq 0$.

So $L(\lambda)$ becomes a nonzero G-module when

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Dimensions of global sections

Using Weyl's dimension formula for irreducible G-modules,

$$\dim(H^0(G/B; \mathcal{L}_\lambda)) = \prod_{1\le i<j\le 2} \frac{(\lambda_i+n-i)-(\lambda_j+n-j)}{(n-i)-(n-j)}$$

$$= \prod_{\text{boxes in } \lambda} \frac{n+c(\text{box})}{h(\text{box})}.$$

For partial flag varieties Fl_μ,

$$\lambda = (\underbrace{\lambda_1,\lambda_1,\ldots,\lambda_1}_{\mu_1}, \underbrace{\lambda_2,\lambda_2,\ldots,\lambda_2}_{\mu_2}, \ldots, \underbrace{\lambda_\ell,\ldots,\lambda_\ell}_{\mu})$$

with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell$.

In particular,

$$\dim(H^0(\mathbb{P}^1, \mathcal{L}_{(\lambda_1,\lambda_2)})) = \dim(H^0(GL_2(\mathbb{C})/B, \mathcal{L}_{(\lambda_1,\lambda_2)}))$$

$$= \prod_{1\le i<j\le 2} \frac{(\lambda_i+n-i)-(\lambda_j+n-j)}{(n-i)-(n-j)} = \frac{(\lambda_1+2-1)-(\lambda_2+2-2)}{(2-1)-(2-2)}$$

$$= \frac{\lambda_1-\lambda_2+1}{1} = d+1, \text{ where } d=\lambda_1-\lambda_2.$$

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G-modules give trivial bundles on G/B

Proposition If V is a G-module then

$$G\times_B V \cong G/B \times V, \text{ a trivial bundle on } G/B$$

of rank $\dim(V)$.

Proof Define maps of vector bundles

$$\begin{array}{ccc} G/B\times V \xrightarrow{\Psi} G\times_B V & \text{and} & G\times_B V \xrightarrow{\Phi} G/B\times V \\ (g,v) \longmapsto [g, g^{-1}v] & & [g,v] \longmapsto (g, gv) \end{array}$$

These are well defined since

$$\Psi(gb,v) = [gb, (gb)^{-1}v] = [gb, b^{-1}g^{-1}v] = [g, g^{-1}v] = \Psi(g,v),$$

$$\Phi([gb,v]) = (gb, gbv) = (g, gbv) = \Phi([g,bv]),$$

and

$$\Psi(\Phi([g,v])) = \Psi(g,gv) = [g, g^{-1}gv] = [g,v]$$

$$\Phi(\Psi(g,v)) = \Phi([g,g^{-1}v]) = (g, gg^{-1}v) = (g,v)$$

so that Φ and Ψ are inverses of each other.

Since

$$\begin{tikzcd} G/B\times V \arrow[r, "\Psi"] \arrow[d, "pr_1"'] & G\times_B V \arrow[d, "\pi"] \\ G/B \arrow[r, "id"] & G/B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G\times_B V \arrow[r, "\Phi"] \arrow[d, "\pi"'] & G/B\times V \arrow[d, "pr_1"] \\ G/B \arrow[r, "id"] & G/B \end{tikzcd}$$

commute then Ψ and Φ are maps of vector bundles.

Hence $G\times_B V \cong G/B \times V$ and $G\times_B V$ is a trivial bundle on G/B. //.

Writing $K(G/B)$ as a quotient of a Laurent polynomial ring.

Let V be a G-module.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$. The <u>$\lambda$-weight space of V</u> is

$$V_\lambda = \left\{ v \in V \;\middle|\; \begin{array}{l} \text{if } h \in T \text{ then} \\ hv = X^\lambda(h) v \end{array} \right\} \cong \mathbb{C}_\lambda^{\oplus m_\lambda} \text{ with } m_\lambda = \dim(V_\lambda).$$

So, as T-modules

$$V = \bigoplus_{\lambda \in \mathbb{Z}^n} V_\lambda = \bigoplus_{\lambda \in \mathbb{Z}^n} (\mathbb{C}_\lambda)^{\oplus m_\lambda}.$$

So

$$[G \times_B V] = \Big[\bigoplus_{\lambda \in \mathbb{Z}^n} (G \times_B \mathbb{C}_\lambda)^{\oplus m_\lambda} \Big] = \bigoplus_{\lambda \in \mathbb{Z}^n} m_\lambda [\mathcal{L}_\lambda]$$

in $K(G/B)$. Let

$$X^\lambda = [\mathcal{L}_\lambda] \text{ in } K(G/B), \text{ so that}$$

$$X^\lambda = X_1^{\lambda_1} X_2^{\lambda_2} \cdots X_n^{\lambda_n} \quad \text{where} \quad X_i = [\mathcal{L}_{\varepsilon_i}] \text{ with}$$

$$\varepsilon_i = (0, \ldots, 0, \underset{i\text{th}}{1}, 0, \ldots 0)$$

Then

$$\frac{\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]}{\left\langle \sum_{\lambda \in \mathbb{Z}^n} m_\lambda X^\lambda = \dim(V) \;\middle|\; \begin{array}{c}\text{for } G\text{-modules} \\ V\end{array} \right\rangle} \xrightarrow{\varphi} K(G/B)$$

$$X^\lambda \longmapsto [\mathcal{L}_\lambda]$$

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(4)

Covering G/B by affine charts

An open cover of G/B is $\mathcal{S} = \{\mathcal{U}_w \mid w \in S_n\}$,

where $\mathcal{U}_w = ww_0(Bw_0B) = ww_0\mathcal{U}^+w_0B = w\mathcal{U}^-B$,

with $\mathcal{U}^+ = \left\{\begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ 0 & & 1\end{pmatrix} \middle| u_{ij} \in \mathbb{C}\right\}$, $\mathcal{U}^- = \left\{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ v_{ij} & & 1\end{pmatrix} \middle| v_{ij} \in \mathbb{C}\right\}$, $w_0 = \begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0\end{pmatrix}$

We know that $Bw_0B = \mathbb{C}^{\binom{n}{2}}$, so $\mathcal{U}_w \cong \mathbb{C}^{\binom{n}{2}}$.

<u>Example</u> If $G = GL_3(\mathbb{C})$ and $B = \left\{\begin{pmatrix} x_1 & u_{12} & u_{13} \\ 0 & x_2 & u_{23} \\ 0 & 0 & x_3\end{pmatrix} \middle| \begin{matrix} x_i \in \mathbb{C}^\times \\ u_{ij} \in \mathbb{C}\end{matrix}\right\}$

then

$$\mathcal{U}_1 = \left\{\begin{pmatrix} a_{11} & a_{12} & 1 \\ a_{21} & 1 & 0 \\ 1 & 0 & 0\end{pmatrix}B \middle| a_{ij} \in \mathbb{C}\right\}$$

$$\mathcal{U}_{s_1} = s_1\mathcal{U}_1 = \left\{\begin{pmatrix} a_{21} & 1 & 0 \\ a_{11} & a_{12} & 1 \\ 1 & 0 & 0\end{pmatrix}B \middle| a_{ij} \in \mathbb{C}\right\}$$

$$\mathcal{U}_{s_2} = s_2\mathcal{U}_1 = \left\{\begin{pmatrix} a_{11} & a_{12} & 1 \\ 1 & 0 & 0 \\ a_{21} & 1 & 0\end{pmatrix}B \middle| a_{ij} \in \mathbb{C}\right\}$$

$$\mathcal{U}_{s_1s_2} = \left\{\begin{pmatrix} 1 & 0 & 0 \\ a_{11} & a_{12} & 1 \\ a_{21} & 1 & 0\end{pmatrix}B \middle| a_{ij} \in \mathbb{C}\right\}$$

$$\mathcal{U}_{s_2s_1} = s_2s_1\mathcal{U}_1 = \left\{\begin{pmatrix} a_{21} & 1 & 0 \\ 1 & 0 & 0 \\ a_{11} & a_{12} & 1\end{pmatrix}B \middle| a_{ij} \in \mathbb{C}\right\}$$

$$U_{s_1s_2s_1} = s_1s_2s_1 U_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{11} & a_{12} & 1 \end{pmatrix} \middle| \ a_{ij} \in \mathbb{C} \right\}$$

The transition maps between charts are determined by the following equalities:

$$\begin{pmatrix} a_{21} & 1 & 0 \\ a_{11} & a_{12} & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{21} & a_{12}^{-1} & 1 \\ a_{11} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{12} & -1 \\ 0 & 0 & a_{12}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & 1 \\ 1 & 0 & 0 \\ a_{21} & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11}a_{21}^{-1} & a_{12}a_{21}+a_{11} & 1 \\ a_{21}^{-1} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{21} & -1 & 0 \\ 0 & a_{21}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{11} & a_{12} & 1 \\ a_{21} & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{21}^{-1} & 1 & 0 \\ a_{11}a_{21}^{-1} & a_{12}a_{21}+a_{11} & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{21} & -1 & 0 \\ 0 & a_{21}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{21} & 1 & 0 \\ 1 & 0 & 0 \\ a_{11} & a_{12} & 1 \end{pmatrix} = \begin{pmatrix} a_{21} & a_{12}^{-1} & 1 \\ 1 & 0 & 0 \\ a_{11} & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{12} & -1 \\ 0 & 0 & a_{12}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{11} & a_{12} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{12}^{-1} & 1 \\ a_{11} & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{12} & -1 \\ 0 & 0 & a_{12}^{-1} \end{pmatrix}$$

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Using the transition maps compute the value of $s_0 : G \to \mathbb{C}$ on each chart. Recall

$$s_0\begin{pmatrix} a_{11} & a_{12} & 1 \\ a_{21} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1 \quad \text{and} \quad s_0(gb) = s_0(g)\,\chi^\lambda(b).$$

So

$$s_0\begin{pmatrix} a_{21} & 1 & 0 \\ a_{11} & a_{12} & 1 \\ 1 & 0 & 0 \end{pmatrix} = s_0\begin{pmatrix} a_{21} & a_{12}^{-1} & 1 \\ a_{11} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \chi^\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{12} & -1 \\ 0 & 0 & a_{12}^{-1} \end{pmatrix}$$

$$= 1 \cdot a_{12}^{\lambda_2-\lambda_3} = a_{12}^{\lambda_2-\lambda_3}$$

$$s_0\begin{pmatrix} a_{11} & a_{12} & 1 \\ 1 & 0 & 0 \\ a_{21} & 1 & 0 \end{pmatrix} = 1 \cdot \chi^\lambda\begin{pmatrix} a_{21} & -1 & 0 \\ 0 & a_{21}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = a_{21}^{\lambda_1-\lambda_2}$$

$$s_0\begin{pmatrix} 1 & 0 & 0 \\ a_{11} & a_{12} & 1 \\ a_{21} & 1 & 0 \end{pmatrix} = s_0\begin{pmatrix} a_{21}^{-1} & 1 & 0 \\ a_{11}a_{21}^{-1} & a_{12}a_{21}+a_{11} & 1 \\ 1 & 0 & 0 \end{pmatrix} a_{21}^{\lambda_1-\lambda_2}$$

$$= s_0\begin{pmatrix} a_{21}^{-1} & (a_{12}a_{21}+a_{11})^{-1} & 1 \\ a_{11}a_{21}^{-1} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} (a_{12}a_{21}+a_{21})^{\lambda_2-\lambda_3} a_{21}^{\lambda_1-\lambda_2}$$

$$= a_{12}^{\lambda_2-\lambda_3} a_{21}^{\lambda_1-\lambda_3} + a_{21}^{\lambda_1-\lambda_3}$$

$$s_0\begin{pmatrix} a_{21} & 1 & 0 \\ 1 & 0 & 0 \\ a_{11} & a_{12} & 1 \end{pmatrix} = s_0\begin{pmatrix} a_{21} & a_{12}^{-1} & \cancel{0} \\ 1 & 0 & 0 \\ a_{11} & 1 & 0 \end{pmatrix} a_{12}^{\lambda_2-\lambda_3}$$

$$= s_0\begin{pmatrix} a_{21}a_{11}^{-1} & a_{12}^{-1}a_{11}+a_{21} & 1 \\ a_{11}^{-1} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} a_{11}^{\lambda_1-\lambda_2} a_{12}^{\lambda_2-\lambda_3} = a_{11}^{\lambda_1-\lambda_2} a_{12}^{\lambda_2-\lambda_3}$$

$$s_0\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{11} & a_{12} & 1 \end{pmatrix} = s_0 \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{12}^{-1} & 1 \\ a_{11} & 1 & 0 \end{pmatrix} a_{12}^{\lambda_2-\lambda_3}$$

$$= s_0 \begin{pmatrix} a_{11}^{-1} & 1 & 0 \\ a_{21}a_{12} & a_{12}^{-1}a_{11}+a_{21} & 1 \\ 1 & 0 & 0 \end{pmatrix} a_{11}^{\lambda_1-\lambda_2} a_{12}^{\lambda_2-\lambda_3}$$

$$= s_0 \begin{pmatrix} a_{11}^{-1} & (a_{12}^{-1}a_{11}+a_{21})^{-1} & 1 \\ a_{21}a_{12} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} (a_{12}^{-1}a_{11}+a_{21})^{\lambda_2-\lambda_3} a_{11}^{\lambda_1-\lambda_2} a_{12}^{\lambda_2-\lambda_3}$$

$$= a_{11}^{\lambda_1-\lambda_3} + a_{11}^{\lambda_1-\lambda_2} a_{21}^{\lambda_2-\lambda_3} a_{12}^{\lambda_2-\lambda_3}$$

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If $s_\gamma^w : G \to \mathbb{C}$ is given by

$$s_\gamma^w\left(w \begin{pmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} w_0 \right) = \prod_{1\le i<j\le n} u_{ij}^{\gamma_{ij}} \in \mathbb{C}[u_{ij}]$$

then

$$\begin{aligned}
h\, s_\gamma^w(w u^+ w_0) &= s_\gamma^w(h^{-1} w u^+ w_0) \\
&= s_\gamma^w(w(w^{-1}h^{-1}w) u^+ (w^{-1}hw)(w^{-1}h^{-1}w) w_0) \\
&= s_\gamma^w(w ((w^{-1}h^{-1}w) u^+ (w^{-1}hw)) w_0 (w_0 w^{-1} h^{-1} w w_0)) \\
&= s_\gamma^w(w((w^{-1}h^{-1}w) u^+ (w^{-1}hw)) w_0) X^\lambda(w_0 w^{-1} h^{-1} w w_0) \\
&= s_\gamma^w(w u^+ w_0) \prod_{1\le i<j\le n} X^{w \sum \gamma_{ij}\alpha_{ij}} X^{-w_0 w^{-1}\lambda} \\
&= X^{-w_0 w^{-1}\lambda + w^{-1}\sum_{1\le i<j\le n} \gamma_{ij}\alpha_{ij}}\, s_\gamma^w(w u^{-1} w_0).
\end{aligned}$$

So the polynomial functions on $w U^+ w_0 B = U_w$ have weight in the cone

$$-w_0 w^{-1}\lambda + w^{-1} Q^+ = w(-w_0\lambda + Q^+).$$

Then

$$\bigcap_{w\in W} w(-w_0\lambda + Q^+) \neq \emptyset$$

if and only if λ is dominant.