Algebraic Geometry Week 10

Riemann-Roch (following Macdonald Ch. 10)

Grothendieck-Riemann-Roch [Harder §9.7.4]

Let $f: X \to Y$ be proper. Let TX be the tangent bundle of X and TY the tangent bundle of Y. Then

$$f_*(Td(T^*X)\,ch(M)) = Td(T^*Y)\,ch(f_! M)$$

$$\begin{tikzcd}
K(X) \arrow[r, "Td(TX)\,ch"] \arrow[d, "f_!"'] & A(X) \otimes \mathbb{Q} \arrow[d, "f_*"] \\
K(Y) \arrow[r, "Td(TY)\,ch"'] & A(Y) \otimes \mathbb{Q}
\end{tikzcd}$$

Hirzebruch-Riemann-Roch

$$\chi(D) = \underset{\text{component}}{\text{degree } d}\left(ch(\mathcal{O}(D))\,Td(TX)\right)$$

Riemann-Roch for surfaces

$$\chi(D) = \tfrac{1}{2} D(D-K) + \chi(X).$$

Riemann-Roch for curves [Harder Theorem 5.1.12]

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g$$

<u>Grothendieck groups</u> [Harder §9.5.4]

Let $\mathcal{C}$ be an abelian category. The <u>Grothendieck group of $\mathcal{C}$</u> $K(\mathcal{C})$ is the abelian group generated by symbols $[M]$ for $M \in \mathcal{C}$, with relations

$$[M]=[N] \text{ if } M \cong N, \text{ and}$$

$[M]=[N]+[P]$ if there exists an exact sequence

$$0 \to N \to M \to P \to 0.$$

Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space. Let

$\mathrm{lf}\mathcal{O}_X$-mod be the category of locally free $\mathcal{O}_X$-modules (i.e. vector bundles on X),

$\mathrm{fg}\mathcal{O}_X$-mod be the category of locally finitely generated $\mathcal{O}_X$-modules (i.e. coherent sheaves).

Define

$$K^{vb}(X) = K(\mathrm{lf}\mathcal{O}_X\text{-mod}) \text{ and } K_{coh}(X) = K(\mathrm{fg}\mathcal{O}_X\text{-mod}).$$

<u>Theorem</u> If X is smooth and projective then

$$K^{vb}(X) \xrightarrow{\sim} K_{coh}(X) \qquad \text{(Poincaré duality)}$$

If X is smooth and projective define

$$K(X) = K^{vb}(X) = K_{coh}(X).$$

Products, pullbacks and pushforwards

Define a product on $K^{vb}(X)$ by

$$[M][N] = [M \otimes N].$$

Then $K^{vb}(X)$ is a commutative ring with $1 = [\mathcal{O}_X]$.

The same operation

$$\begin{array}{rcl} K^{vb}(X) \times K_{coh}(X) & \longrightarrow & K_{coh}(X) \\ ([M],[N]) & \longmapsto & [M \otimes N] \end{array}$$

makes $K_{coh}(X)$ into a $K^{vb}(X)$-module. There is always a map

$$\begin{array}{rcl} K^{vb}(X) & \longrightarrow & K_{coh}(X) \\ [M] & \longmapsto & [M]. \end{array}$$

Let $f: X \to Y$ be a morphism. The <u>pullback</u> is the ring homomorphism

$$\begin{array}{rcl} f^*: K^{vb}(Y) & \longrightarrow & K^{vb}(X) \\ [M] & \longmapsto & [f^*M]. \end{array}$$

Let $f: X \to Y$ be a proper morphism. The <u>pushforward</u> is the $K^{vb}(Y)$-module homomorphism

$f_!: K_{coh}(X) \longrightarrow K_{coh}(Y)$ given by

$$f_![M] = \sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j [R^j f_* M].$$

The Chow ring $A(X)$ [Harder §9.7.3]

Let X be smooth irreducible and projective

$$d = \dim(X).$$

Let $A^k(X)$ be the abelian group generated by symbols $[D]$, with D an irreducible closed subvariety of codimension k with relations

$[D_0] = [D_1]$ if there exists $C \subseteq X \times \mathbb{A}^1$ such that

$$C \cap (X \times \{0\}) = D_0 \times \{0\} \text{ and } C \cap (X \times \{1\}) = D_1 \times \{1\}.$$

Define a graded ring

$$A(X) = \bigoplus_{k=0}^{d} A^k(X) \quad \text{with} \quad [Z_1][Z_2] = [Z_1 \cap Z_2]$$

where Z_1 and Z_2 are representatives of the equivalence class which intersect properly.

Let $f: X \to Y$ be a morphism. The <u>pullback</u> is the graded ring homomorphism

$$f^*: A(Y) \longrightarrow A(X) \quad \text{given by} \quad f^*[D] = [f^{-1}(D)]$$

Let $f: X \to Y$ be a proper morphism. The <u>pushforward</u> is the additive (but not multiplicative and not graded)

$$f_*: A(X) \longrightarrow A(Y) \quad \text{given by} \quad f_*[D] = [f(D)].$$

The <u>projection formula</u> is

$$f_*(c \cdot f^*(D)) = f_*(c) \cdot D, \quad \text{for } c \in A(X),\ D \in A(Y).$$

The Chern character $ch: K(X) \to A(X) \otimes \mathbb{Q}$

The Chern character is the ring homomorphism

$$ch: K(X) \longrightarrow A(X) \otimes \mathbb{Q}$$

satisfying

(a) If $f: X \to Y$ is a morphism then

$$ch(f^*M) = f^*(ch\, M).$$

(b) If $D \in A^1(X)$ then

$$ch(\mathcal{O}(D)) = 1 + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots$$

The Chern roots of M are $\gamma_1, \ldots, \gamma_n$ given by

$$ch(M) = e^{\gamma_1} + e^{\gamma_2} + \cdots + e^{\gamma_n}, \quad \text{where } n = \mathrm{rank}(M).$$

The Chern classes of M are $c_1(M), \ldots, c_n(M)$ given by

$$1 + c_1(M)t + \cdots + c_n(M)t^n = (1+\gamma_1 t)\cdots(1+\gamma_n t).$$

The Todd class of M is

$$Td(M) = \left(\frac{\gamma_1}{1-e^{-\gamma_1}}\right)\left(\frac{\gamma_2}{1-e^{-\gamma_2}}\right)\cdots\left(\frac{\gamma_n}{1-e^{-\gamma_n}}\right)$$

Alg. Geom. Week 10

Line bundle $\mathcal{O}(D)$ of a divisor D

A divisor is an element $D \in A^1(X)$,

$$D = n_1[D_1] + n_2[D_2] + \cdots + n_\ell[D_\ell] \quad \text{with } n_1, \ldots, n_\ell \in \mathbb{Z}.$$

Let $\mathcal{S}$ be an open cover of X.
The line bundle $\mathcal{O}(D)$ is determined by

$$(g_{UV})_{U,V \in \mathcal{S}} \quad \text{where} \quad g_{UV} : U \cap V \to \mathbb{C}^\times$$

is given by

$$g_{UV} = h_U h_V^{-1}$$

where

$$h_U = f_{1U}^{n_1} f_{2U}^{n_2} \cdots f_{\ell U}^{n_\ell} \quad \text{with } f_{iU} \in \mathcal{O}_X(U)$$

such that

$$D_i = \{x \in U \mid f_{iU}(x) = 0\}$$

Conversely, given a line bundle one can reconstruct the divisor (see [Harder §9.4 p. 194-195])

Let TX be the tangent bundle of X.
The canonical divisor is

$$K = [\Lambda^d TX], \quad \text{where } d = \dim X.$$

Grothendieck RR & Hirzebruch RR

Let $f: X \to pt$, with $d = \dim X$.

Since a vector bundle V on pt is a vector space

$$\begin{matrix} ch: K(pt) & \xrightarrow{\sim} & \mathbb{Z}[pt] = A(pt) \\ V & \longmapsto & \dim(V) \end{matrix}$$

Since $A^0(pt) = \mathbb{Z}$ and $A^i(pt) = 0$, for $i \in \mathbb{Z}_{>0}$ then

$f_*: A(X) \to A(pt) = \mathbb{Z}[pt]$ is

$$\begin{aligned} f_*(D) &= \text{degree } d \text{ component of } D. \\ &= \text{highest degree term of } D \\ &= \text{coeff. of } [X] \text{ in } D. \end{aligned}$$

Since $f_*: Sh(X) \to Sh(pt)$ is $H^0(X, -)$ and $Td(T_{pt}) = 1$,

$$\begin{aligned} Td(T_{pt})\, ch(f_! M) &= 1 \cdot ch\Big(\sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j R^j f_* M\Big) \\ &= ch\Big(\sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j R^j H^0(X, M)\Big) \\ &= ch\Big(\sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j H^j(X, M)\Big) \\ &= \sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j \dim H^j(X, M) = \chi(X, M). \end{aligned}$$

So the lefthand side of GRR is $\chi(X, M)$.
which is the left hand side of HRR.
The right hand side of GRR is

$$f_*(Td(TX)\, ch(M)) = \underset{\text{component}}{\text{degree } d}\, (Td(X)\, ch(M)).$$

HRR to RR for surfaces and curves

First apply HRR to $D=0$. Since $\mathcal{O}(0)=\mathcal{O}_X$ then
If $\dim(X)=1$ then

$$\chi(X) = \underset{\text{component}}{\deg 1}\left(ch(\mathcal{O}(0))\,Td(T^*X)\right)$$

$$= \underset{\text{component}}{\deg 1}\left((1+0)\cdot \frac{\gamma_1}{1-e^{-\gamma_1}}\right)$$

$$= \underset{\text{component}}{\deg 1}\left(1\cdot(1+\tfrac{1}{2}\gamma_1)\right) = \tfrac{1}{2}\gamma_1 = 1-g.$$

If $\dim(X)=2$ then

$$\chi(X) = \underset{\text{component}}{\deg 2}\left(ch(\mathcal{O}(0))\,Td(T^*X)\right)$$

$$= \underset{\text{component}}{\deg 2}\left(\left(1+0+\frac{0^2}{2}\right)\left(\frac{\gamma_1}{1-e^{-\gamma_1}}\right)\left(\frac{\gamma_2}{1-e^{-\gamma_2}}\right)\right)$$

$$= \underset{\text{component}}{\deg 2}\left(1\cdot\left(1+\tfrac{1}{2}\gamma_1+\tfrac{1}{12}\gamma_1^2\right)\left(1+\tfrac{1}{2}\gamma_2+\tfrac{1}{12}\gamma_2^2\right)\right)$$

$$= \tfrac{1}{12}(\gamma_1^2+\gamma_2^2)+\tfrac{1}{4}\gamma_1\gamma_2 = \tfrac{1}{12}(c_1^2+c_2),$$

where $c_1 = c_1(T^*X)$ and $c_2 = c_2(T^*X)$.

Now apply HRR to a general divisor D. Let

$$\lambda = c_1(\mathcal{O}(D)).$$

If $\dim(X)=1$ then

$$\chi(D) = \underset{\text{comp.}}{\deg 1}\left(ch(\mathcal{O}(D))\cdot Td(T^*X)\right)$$

$$= \underset{\text{comp}}{\deg 1}\left((1+\lambda)\cdot(1+\tfrac{1}{2}\gamma_1)\right) = \lambda+\tfrac{1}{2}\gamma_1 = \deg(\mathcal{O}(D))+1-g$$

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If $\dim(X)=2$ then

$$\chi(D) = \deg_{\text{comp.}} 2\left((1+\lambda+\tfrac{1}{2}\lambda^2)\cdot Td(T^*X)\right)$$

$$= \deg_{\text{comp}} 2\left((1+\lambda+\tfrac{1}{2}\lambda^2)(1+\tfrac{1}{2}c_1+\tfrac{1}{12}(c_1^2+c_2))\right)$$

$$= \tfrac{1}{12}(c_1^2+c_2) + \tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\lambda c_1$$

$$= \tfrac{1}{2}\lambda(\lambda+c_1) + \chi(X)$$

$$= \tfrac{1}{2}D(D-K) + \chi(X),$$

since c_1 is the class of $-K$.

Alg. Geom. Week 10

Chow ring, cohomology and Projection formulas

(10)

[Harder §9.7.3 p 248] Define a homomorphism

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$$A^k(X) \longrightarrow H^{2k}(X, \mathbb{Z})$$
$$[Z] \longmapsto [Z]$$

Where $[Z]$ in $H^{2k}(X, \mathbb{Z})$ is the fundamental class of Z

Projection formula for K-theory

Let $f: X \to Y$ be a proper morphism.

Let $N \in K^{vb}(Y)$ and $M \in K_{coh}(X)$. Then

$$f_!(f^*(N)M) = N(f_! M)$$

Projection formula for the Chow ring

Let $f: X \to Y$ be a proper morphism.

Let $\nu \in A(Y)$ and $\mu \in A(X)$. Then

$$f_*(f^*(\nu)\mu) = \nu(f_*\mu)$$