recd 10.08.2018 2pm

$\frac{14}{15} + \frac{5}{5} = \frac{19}{20}$

Algebraic Geometry
Assignment 1

Tom Dove

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Part 1

(a) topological space

A **topological space** is a set X and a collection $\mathcal{T}$ of subsets of X such that:

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\bigcup_{V \in \mathcal{S}} V \in \mathcal{T}$.

(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, ..., U_\ell \in \mathcal{T}$ then $U_1 \cap ... \cap U_\ell \in \mathcal{T}$.

Example: Let $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$.

To show: $(X, \mathcal{T})$ is a topological space.

To show:

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\bigcup_{V \in \mathcal{S}} V \in \mathcal{T}$.

(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, ..., U_\ell \in \mathcal{T}$ then $U_1 \cap ... \cap U_\ell \in \mathcal{T}$.

(a) This is true by inspection.

(b) Assume $\mathcal{S} \subset \mathcal{T}$.

To show: $\bigcup_{V \in \mathcal{S}} V \in \mathcal{T}$

If $\mathcal{S}$ contains just one open set then this is true. We consider the four other cases:

- Case 1: $\mathcal{S} = \{\emptyset, \{a\}\}$. In this case $\bigcup_{V \in \mathcal{S}} V = \emptyset \cup \{a\} = \{a\} \in \mathcal{T}$.
- Case 2: $\mathcal{S} = \{\emptyset, \{a, b\}\}$. In this case $\bigcup_{V \in \mathcal{S}} V = \emptyset \cup \{a, b\} = \{a, b\} \in \mathcal{T}$.
- Case 3: $\mathcal{S} = \{\{a\}, \{a, b\}\}$. In this case $\bigcup_{V \in \mathcal{S}} V = \{a\} \cup \{a, b\} = \{a, b\} \in \mathcal{T}$.

- Case 4: $\mathcal{S} = \{\emptyset, \{a\}, \{a,b\}\}$. In this case $\bigcup_{V\in\mathcal{S}} V = \emptyset \cup \{a\} \cup \{a,b\} = \{a,b\} \in \mathcal{T}$.

Thus in every case $\bigcup_{V\in\mathcal{S}} V \in \mathcal{T}$.

(c) Assume $\ell \in \mathbb{Z}_{>0}$ and $U_1, ..., U_\ell \in \mathcal{T}$.

To show: $U_1 \cap ... \cap U_\ell \in \mathcal{T}$.

If $\ell = 1$ then this is true. We consider the four other cases:

- Case 1: $\ell = 2$ with $U_1 = \emptyset$ and $U_2 = \{a\}$. In this case $U_1 \cap U_2 = \emptyset \in \mathcal{T}$.
- Case 2: $\ell = 2$ with $U_1 = \emptyset$ and $U_2 = \{a,b\}$. In this case $U_1 \cap U_2 = \emptyset \in \mathcal{T}$.
- Case 3: $\ell = 2$ with $U_1 = \{a\}$ and $U_2 = \{a,b\}$. In this case $U_1 \cap U_2 = \{a\} \in \mathcal{T}$.
- Case 4: $\ell = 3$ with $U_1 = \emptyset$, $U_2 = \{a\}$ and $U_3 = \{a,b\}$. In this case $U_1 \cap U_2 \cap U_3 = \emptyset \in \mathcal{T}$.

Thus in every case $U_1 \cap ... \cap U_\ell \in \mathcal{T}$.

So $(X, \mathcal{T})$ is a topological space.

(b) metric space

A **metric space** (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that:

(a) If $x \in X$ then $d(x,x) = 0$.

(b) If $x, y \in X$ and $d(x,y) = 0$ then $x = y$.

(c) If $x, y \in$ then $d(x,y) = d(y,x)$.

(d) If $x, y, z \in X$ then $d(x,y) \leq d(x,z) + d(y,z)$.

Example: Let $C[0,1]$ be the set of continuous functions from $\mathbb{R}_{[0,1]}$ to $\mathbb{R}$. Let $d : C[0,1] \times C[0,1] \to \mathbb{R}_{\geq 0}$ be the function

$$d(f,g) = \sup_{x\in\mathbb{R}_{[0,1]}} |f(x) - g(x)|.$$

To show: $(C[0,1], d)$ is a metric space.

To show:

(a) If $f \in C[0,1]$ then $d(f,f) = 0$.

(b) If $f, g \in C[0,1]$ and $d(f,g) = 0$ then $f = g$.

(c) If $f, g \in C[0,1]$ then $d(f,g) = d(g,f)$.

(d) If $f, g, h \in C[0,1]$ then $d(f,g) \leq d(f,h) + d(g,h)$.

(a) Assume $f \in C[0,1]$.

To show: $d(f,f) = 0$.

$$d(f,f) = \sup_{x \in \mathbb{R}_{[0,1]}} |f(x) - f(x)| = \sup_{x \in \mathbb{R}_{[0,1]}} |0| = 0.$$

(b) Assume $f, g \in C[0,1]$ and $d(f,g) = 0$

To show: $f = g$.

We know that:

$$0 = d(f,g) = \sup_{x \in \mathbb{R}_{[0,1]}} |f(x) - g(x)|$$

So $\{|f(x) - g(x)| : x \in \mathbb{R}_{[0,1]}\}$ is a set of non-negative real numbers that is bounded above by 0. This implies that $|f(x) - g(x)| = 0$ for all $x \in \mathbb{R}_{[0,1]}$ and so $f = g$.

(c) Assume $f, g \in C[0,1]$.

To show: $d(f,g) = d(g,f)$.

$$d(f,g) = \sup_{x \in \mathbb{R}_{[0,1]}} |f(x) - g(x)| = \sup_{x \in \mathbb{R}_{[0,1]}} |g(x) - f(x)| = d(g,f).$$

(d) Assume $f, g, h \in C[0,1]$.

To show: $d(f,g) \le d(f,h) + d(g,h)$.

Assuming the standard triangle inequality on $\mathbb{R}$,

$$\begin{aligned} d(f,g) &= \sup_{x \in \mathbb{R}_{[0,1]}} |f(x) - g(x)| \\ &= \sup_{x \in \mathbb{R}_{[0,1]}} |f(x) - h(x) + h(x) - g(x)| \\ &\le \sup_{x \in \mathbb{R}_{[0,1]}} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\le \sup_{x \in \mathbb{R}_{[0,1]}} |f(x) - h(x)| + \sup_{x \in \mathbb{R}_{[0,1]}} |h(x) - g(x)| \\ &= d(f,h) + d(h,g) \end{aligned}$$

do these need justification? What is the definition of sup.

So $(C[0,1], d)$ is a metric space.

(c) ringed space

A **ringed space** $(X, \mathcal{T}_X, \mathcal{O}_X)$ is a topological space $(X, \mathcal{T}_X)$ with a sheaf of rings $\mathcal{O}_X$ on X.

does this need defn?

Example: Let $(X, \mathcal{T}_X)$ be some topological space and for $U \in \mathcal{T}_X$ define $\mathcal{O}_X(U)$ to be the ring of continuous functions on U.

$$\begin{aligned} \mathcal{O}_X : \mathcal{T} &\to \{\text{commutative rings with identity}\} \\ U &\mapsto \{f : U \to \mathbb{R} \mid f \text{ continuous}\} \\ \iota_U^V &\mapsto (\mathrm{res}_U^V : \mathcal{O}_X(V) \to \mathcal{O}_X(U)) \end{aligned}$$

where $\iota_U^V : U \to V$ is the inclusion of U into V and res_U^V is the restriction map $\mathrm{res}_U^V(f) = f|_V$.

To show: $(X, \mathcal{T}_X, \mathcal{O}_X)$ is a ringed space.

To show: $\mathcal{O}_X$ is a sheaf of rings on X.

To show:

(a) $\mathcal{O}_X$ is a pre-sheaf of rings on X.

is this really the axiom for a sheaf?

(b) If $U \in \mathcal{T}_X$ and $\{U_\alpha\}_{\alpha \in A}$ is an open cover of U and if $f \in \mathcal{O}_X(U)$ such that $res_{U_\alpha}^U(f) = 0$ for all $\alpha \in A$ then $f = 0$.

(c) If $U \in \mathcal{T}_X$ and $\{U_\alpha\}_{\alpha \in A}$ is an open cover of U and if for all pairs $f_\alpha \in \mathcal{O}_X(U_\alpha)$ and $f_\beta \in \mathcal{O}_X(U_\beta)$ we have $\mathrm{res}_{U_\alpha \cap U_\beta}^{U_\alpha}(f_\alpha) = \mathrm{res}_{U_\alpha \cap U_\beta}^{U_\beta}(f_\beta)$ then there exists $f \in \mathcal{O}_X(U)$ such that $\mathrm{res}_{U_\alpha}^U(f) = f_\alpha$ and $\mathrm{res}_{U_\beta}^U(f) = f_\beta$.

(a) **To show:**

- (aa) If $U, V, W \in \mathcal{T}_X$ and $U \subseteq V \subseteq W$ then $\mathrm{res}_U^W = \mathrm{res}_U^V \circ \mathrm{res}_V^W$.
- (ab) If $f \in \mathcal{O}_X(U)$ then $\mathrm{res}_U^U(f) = f$.

(aa) Assume $U, V, W \in \mathcal{T}_X$ and $U \subseteq V \subseteq W$.

To show: $\mathrm{res}_U^W = \mathrm{res}_U^V \circ \mathrm{res}_V^W$.

To show: If $f \in \mathcal{O}_X(W)$ then $\mathrm{res}_U^W(f) = \mathrm{res}_U^V \circ \mathrm{res}_V^W(f)$.

Assume $f \in \mathcal{O}_X(W)$.

To show: $\mathrm{res}_U^W(f) = \mathrm{res}_U^V \circ \mathrm{res}_V^W(f)$.

$$\mathrm{res}_U^V \circ \mathrm{res}_V^W(f) = (f|_V)|_U = f|_U = \mathrm{res}_U^W(f)$$

So $\mathrm{res}_U^W = \mathrm{res}_U^V \circ \mathrm{res}_V^W$.

(ab) Assume $f \in \mathcal{O}_X(U)$.

To show: $\mathrm{res}_U^U(f) = f$.

$$\mathrm{res}_U^U(f) = f|_U = f$$

So $\mathcal{O}_X$ is a pre-sheaf of rings.

(b) Assume the following:

- $U \in \mathcal{T}_X$.
- $\{U_\alpha\}_{\alpha \in A}$ is an open covering of U.
- $f \in \mathcal{O}_X(U)$ such that if $\alpha \in A$ then $\mathrm{res}^U_{U_\alpha}(f) = 0$.

To show: $f = 0$.

To show: If $x \in U$ then $f(x) = 0$.

Assume $x \in U$.

To show: $f(x) = 0$.

Choose $\alpha \in A$ such that $x \in U_\alpha$. Then

$$f(x) = f|_{U_\alpha}(x) = \mathrm{res}^U_{U_\alpha}(f)(x) = 0.$$

So $f = 0$.

(c) Assume the following:

- $U \in \mathcal{T}_X$.
- $\mathcal{S} =$ $\{U_\alpha\}_{\alpha \in A}$ is an open covering of U.
- If $\alpha \in A$ there exists a map $f_\alpha \in \mathcal{O}_X(U_\alpha)$ $U_\alpha, U_\beta \in \mathcal{S}$ and $f_\alpha \in \mathcal{O}_X(U_\alpha)$ and $f_\beta \in \mathcal{O}_X(U_\beta)$
- ~~If $\alpha, \beta \in A$ then~~ $\mathrm{res}^{U_\alpha}_{U_\alpha \cap U_\beta}(f_\alpha) = \mathrm{res}^{U_\beta}_{U_\alpha \cap U_\beta}(f_\beta)$.

To show: There exists $f \in \mathcal{O}_X(U)$ such that $\mathrm{res}^U_{U_\alpha}(f) = f_\alpha$ and $\mathrm{res}^U_{U_\beta}(f) = f_\beta$.

Define f as follows: If $x \in U$ then choose $\alpha \in A$ such that $x \in U_\alpha$ and define $f(x)$ to be $f_\alpha(x)$.

To show:

- (ca) f is well defined.
- (cb) If $\alpha \in A$ then $f|_{U_\alpha} = f_\alpha$.
- (cc) f is continuous.

(ca) **To show:** If $x \in U_\alpha \cap U_\beta$ then $f_\alpha(x) = f_\beta(x)$.

This is true by our assumption that $f_\alpha|_{U_\alpha \cap \beta} = f_\beta|_{U_\alpha \cap \beta}$.

(cb) This is true by our construction of f.

(cc) **To show:** If $S \subseteq \mathbb{R}$ is open, with the standard topology on $\mathbb{R}$, then $f^{-1}(S) \subset U$ is open in U.

Assume $S \subseteq \mathbb{R}$ is open.

To show: $f^{-1}(U)$ is open in U.

$$f^{-1}(U) = f^{-1}(\bigcup_{\alpha\in A} U_\alpha) = \bigcup_{\alpha\in A} f^{-1}(U_\alpha) = \bigcup_{\alpha\in A} f_\alpha^{-1}(U_\alpha)$$

So since each f_α is continuous, each $f_\alpha^{-1}(U_\alpha)$ is open and hence $f^{-1}(U)$ is open as the union of open sets.

So f is continuous.

So $\mathcal{O}_X$ is a sheaf of rings on X.

(d) affine space

Let k be an algebraically closed field. **Affine n-space** is the ringed space $(\mathbb{A}^n, \mathcal{T}_{\mathbb{A}^n}, \mathcal{O}_{\mathbb{A}^n})$ where:

- $\mathbb{A}^n = k^n$.
- $\mathcal{T}_{\mathbb{A}^n}$ is the Zariski topology on $\mathbb{A}^n$, in other words $\mathcal{T}_{\mathbb{A}^n}$ is the topology in which the closed sets are exactly those of the form

$$V(S) = \{(\lambda_1, ..., \lambda_n) \in \mathbb{A}^n \mid \text{if } f \in S \text{ then } f(\lambda_1, ..., \lambda_n) = 0\}$$

  for subsets $S \subseteq k[x_1, ..., x_n]$. definition?
- $\mathcal{O}_{\mathbb{A}^n}$ is the sheaf on $\mathbb{A}^n$ such that if $U \in \mathcal{T}_{\mathbb{A}^n}$ then $\mathcal{O}_{\mathbb{A}^n}(U)$ is the ring of regular functions on U.

Another definition of affine space [Har, §6.2 Example 6] is: Given a commutative ring R, the n-dimensional affine space over Spec R is Spec$(R[X_1, ..., X_n])$. For the definition of Spec and an example of affine space defined in this way, see part (q). Why?! does this coincide with the above?

Good

Example: I will illustrate that affine space is actually a ringed space by first proving that $\mathcal{T}_{\mathbb{A}^n}$ is a topology on $\mathbb{A}^n$ and then that $\mathcal{O}_{\mathbb{A}^n}$ is actually a sheaf on $\mathbb{A}^n$. Proof that they coincide?

To show: $\mathcal{T}_{\mathbb{A}^n}$ is a topology on $\mathbb{A}^n$.

I use the closed set definition of a topology. View $\mathcal{T}_{\mathbb{A}^n}$ as the set of closed sets in $\mathbb{A}^n$.

To show:

(a) $\emptyset \in \mathcal{T}_{\mathbb{A}^n}$ and $\mathbb{A}^n \in \mathcal{T}_{\mathbb{A}^n}$.

(b) If $\mathcal{S} \subset \mathcal{T}_{\mathbb{A}^n}$ then $\bigcap_{U\in\mathcal{S}} U \in \mathcal{T}_{\mathbb{A}^n}$.

(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, ..., U_\ell \in \mathcal{T}_{\mathbb{A}^n}$ then $U_1 \cup U_2 \cup ... \cup U_\ell \in \mathcal{T}_{\mathbb{A}^n}$.

(a) If $S = 1$ then $\emptyset = V(S) \in \mathcal{T}_{\mathbb{A}^n}$.

If $S = \{0\}$ then $\mathbb{A}^n = V(S) \in \mathcal{T}_{\mathbb{A}^n}$.

(b) Assume $\mathcal{S} \subseteq \mathcal{T}_{\mathbb{A}^n}$. Write $\mathcal{S} = \{V(S_\alpha)\}_{\alpha\in\mathcal{S}}$ where $S_\alpha \subseteq k[x_1, ..., x_n]$.

To show: $\bigcap_{\alpha \in A} V(S_\alpha) \in \mathcal{T}_{\mathbb{A}^n}$.

To show: There exists $S \subseteq k[x_1, ..., x_n]$ such that $\bigcap_{\alpha \in A} V(S_\alpha) = V(S)$.

Define $S = \bigcup_{\alpha \in S} S_\alpha$.

To show: $\bigcap_{\alpha \in A} V(S_\alpha) = V(S)$.

Note that we can write

$$\bigcap_{\alpha \in A} V(S_\alpha) = \{\lambda \in \mathbb{A}^n \mid \text{ if } \alpha \in A \text{ and } f \in S_\alpha \text{ then } f(\lambda) = 0\}$$

which by inspection is equivalent to

$$V(S) = \{\lambda \in \mathbb{A}^n \mid \text{ if } f \in \bigcup_{\alpha \in S} S_\alpha \text{ then } f(\lambda) = 0\}.$$

So $\bigcap_{U \in S} U \in \mathcal{T}_{\mathbb{A}^n}$.

(b) Assume $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, ..., U_\ell \in \mathcal{T}_{\mathbb{A}^n}$. We can write $U_i = V(S_i)$ for $i = 1, ..., \ell$ and $S_i \subseteq k[x_1, ..., x_n]$.

To show: $U_1 \cup U_2 \cup ... \cup U_\ell \in \mathcal{T}_{\mathbb{A}^n}$.

To show: There exists $S \subseteq k[x_1, ..., x_2]$ such that $V(S_1) \cup ... \cup V(S_\ell) = V(S)$.

Let $S = \{p_1 p_2 \cdots p_\ell \in k[x_1, ..., x_n] \mid p_i \in S_i\}$.

To show: $V(S_1) \cup ... \cup V(S_\ell) = V(S)$.

To show:

- (ba) $V(S_1) \cup ... \cup V(S_\ell) \subseteq V(S)$.
- (bb) $V(S_1) \cup ... \cup V(S_\ell) \supseteq V(S)$

(ba) Assume $\lambda \in V(S_1) \cup ... \cup V(S_n)$. This means that there exists $\alpha \in A$ such that if $f \in S_\alpha$ then $f(\lambda) = 0$.

To show: $\lambda \in V(S)$.

To show: If $f \in S$ then $f(\lambda) = 0$.

Assume $f \in S$. Then $f = p_1 p_2 \cdots p_\ell$ for polynomials $p_i \in S_i$. In particular $p_\alpha \in S_\alpha$. Therefore $f(\lambda) = p_1(\lambda) \cdots p_\alpha(\lambda) \cdots p_\ell(\lambda) = 0$.

So $V(S_1) \cup ... \cup V(S_\ell) \subseteq V(S)$.

(bb) Assume $\lambda \in V(S)$.

To show: $\lambda \in V(S_1) \cup ... \cup V(S_\ell)$.

To show: There exists $\alpha \in A$ such that if $f \in S_\alpha$ then $f(\lambda) = 0$.

For the purpose of a contradiction assume no such α exists. Then for each $\alpha \in A$ there exists a polynomial $f_\alpha \in S_\alpha$ such that $f_\alpha(\lambda) \neq 0$. But then $f = f_1 f_2 \cdots f_\ell$ is a polynomial in S such that $f(\lambda) \neq 0$, a contradiction.

So $\lambda \in V(S_1) \cup \ldots \cup V(S_\ell)$.

So $V(S) \subseteq V(S_1) \cup \ldots \cup V(S_\ell)$.

So $U_1 \cup \ldots \cup U_\ell \in \mathcal{T}_{\mathbb{A}^n}$

So $\mathcal{T}_{\mathbb{A}^n}$ is a topology on $\mathbb{A}^n$.

Now I will prove that the sheaf of regular functions on $\mathcal{O}_{\mathbb{A}^n}$ is indeed a sheaf. I will not prove it is a presheaf - this comes from the basic properties of (regular) functions and is seen in my example for a ringed space.

To show: $\mathcal{O}_{\mathbb{A}^n}$ is a sheaf on $\mathbb{A}^n$.

Now and for the rest of the assignment I use the definition of sheaf from [Har77, p.61].

To show:

(a) If $U \in \mathcal{T}_X$ and $\{U_\alpha\}_{\alpha \in A}$ is an open cover of U and if $f \in \mathcal{O}_{\mathbb{A}^n}(U)$ such that $f\,|_{U_\alpha} = 0$ for all $\alpha \in A$ then $f = 0$.

(b) If $U \in \mathcal{T}_X$ and $\{U_\alpha\}_{\alpha \in A}$ is an open cover of U and if for all pairs $f_\alpha \in \mathcal{O}_{\mathbb{A}^n}(U_\alpha)$ and $f_\beta \in \mathcal{O}_{\mathbb{A}^n}(U_\beta)$ we have $f_\alpha\,|_{U_\alpha \cap U_\beta} = f_\beta\,|_{U_\alpha \cap U_\beta}$ then there exists $f \in \mathcal{O}_{\mathbb{A}^n}(U)$ such that $f\,|_{U_\alpha} = f_\alpha$ and $f\,|_{U_\beta} = f_\beta$.

(a) Assume the following:

- $U \in \mathcal{T}_X$.
- $\{U_\alpha\}_{\alpha \in A}$ is an open covering of U.
- $f \in \mathcal{O}_{\mathbb{A}^n}(U)$ such that if $\alpha \in A$ then $f\,|_{U_\alpha} = 0$.

To show: $f = 0$.

To show: If $x \in U$ then $f(x) = 0$.

Assume $x \in U$. Let $\alpha \in A$ be such that $x \in U_\alpha$. This exists since we have an open covering of U.

To show: $f(x) = 0$.

$$f(x) = f\,|_{U_\alpha}(x) = 0.$$

(b) Assume the following:

- $U \in \mathcal{T}_X$.
- $\{U_\alpha\}_{\alpha \in A}$ is an open covering of U.
- For each $\alpha \in A$ we have a map $f_\alpha \in \mathcal{O}_{\mathbb{A}^n}(U_\alpha)$.

- If $\alpha, \beta \in A$ then $f_\alpha |_{U_\alpha \cap U_\beta} = f_\beta |_{U_\alpha \cap U_\beta}$.

To show: There exists $f \in \mathcal{O}_{\mathbb{A}^n}(U)$ such that $f |_{U_\alpha} = f_\alpha$ and $f |_{U_\beta} = f_\beta$.

Define f as follows: For $x \in U$ choose $\alpha \in A$ such that $x \in U_\alpha$ and define $f(x)$ to be $f_\alpha(x)$.

To show:

- (ba) f is well defined.
- (bb) $f \in \mathcal{O}_{\mathbb{A}^n}(U)$.
- (bc) $f |_{U_\alpha} = f_\alpha$ and $f |_{U_\beta} = f_\beta$.

(ba) **To show:** If $x \in U_\alpha \cap U_\beta$ then $f_\alpha(x) = f_\beta(x)$.

This is true by the assumption that $f_\alpha |_{U_\alpha \cap U_\beta} = f_\beta |_{U_\alpha \cap U_\beta}$.

(bb) **To show:** f is a regular function on U.

To show: If $a \in U$ then there exists $p, q \in k[x_1, ..., x_n]$ and a neighbourhood V_a of a such that if $x \in V_a$ then $q(a) \neq 0$ and $f(x) = \frac{p(x)}{q(x)}$.

good defn of regular function.

Assume $a \in U$.

To show: There exists $p, q \in k[x_1, ..., x_n]$ and a neighbourhood V_a of a such that if $x \in V_a$ then $q(a) \neq 0$ and $f(x) = \frac{p(x)}{q(x)}$.

Let $\alpha \in A$ be such that $a \in U_\alpha$. Since $f_\alpha \in \mathcal{O}_{\mathbb{A}^n}(U_\alpha)$ we know that f_α is regular on U_α so there exists a neighbourhood, which we define as V_a, of a and polynomials, which we define as p and q, such that if $x \in V_a$ then $q(x) = 0$ and $f_\alpha(x) = \frac{p(x)}{q(x)}$. By our construction of f, we have $f(x) = f_\alpha(x) = \frac{p(x)}{q(x)}$.

So f is regular on U.

(bc) This is true by definition of f.

So $\mathcal{O}_{\mathbb{A}^n}$ is a sheaf on $\mathbb{A}^n$.

(e) projective space

Let k be an algebraically closed field. **Projective n-space** is the ringed space $(\mathbb{P}^n, \mathcal{T}, \mathcal{O}_{\mathbb{P}^n})$, where

- $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\}) / \sim$ where $\sim$ is the equivalence relation in which $(\lambda_0, ..., \lambda_n) \sim (\lambda'_1, ...\lambda'_n)$ if and only if there exists $c \in k$ such that $(\lambda_0, ..., \lambda_n) = c(\lambda'_1, ...\lambda'_n)$.
- $\mathcal{T}$ is the Zariski topology on $\mathbb{P}^n$, in other words $\mathcal{T}$ is the topology in which the closed sets are exactly those of the form

$$V_P(I) = \{[\lambda_0, ..., \lambda_n] \in \mathbb{P}^n \mid \text{if } f \in S \text{ then } f(\lambda_0, ..., \lambda_n) = 0\}$$

  for subsets $S \subseteq k[x_0, ..., x_n]$ of homogeneous polynomials.

Example? $\mathbb{P}^1$?

definition of this?

- $\mathcal{O}_{\mathbb{P}^n}$ is the sheaf on $\mathbb{P}^n$ such that if $U \in \mathcal{T}$ then $\mathcal{O}_{\mathbb{P}^n}(U)$ is the ring of regular functions on U.

Why is this a scheme? Why is this a variety?

good reference

(f) affine variety

[Ram] Let k be an an algebraically closed field. An affine (algebraic) variety is a set

chapter/verse!

$$X = \{(x_1, ..., x_n) \mid \text{if } f \in S \text{ then } f(x_1, ..., x_n) = 0\}$$

Only a set?

where S is a set of polynomials in $k[t_1, ..., t_n]$. By the definition of the Zariski topology this is equivalent to a closed set in affine space over k.

Example: Curves, surfaces and hyper-surfaces, as defined in parts (t), (u) and (v) are all examples of affine varieties. Really? $\mathbb{P}^1$ is a curve, but I don't think it is an affine variety..

(g) variety

Let k be an algebraically closed field. A **k-variety** is a ringed space $(X, \mathcal{T}_x, \mathcal{O}_X)$ that is locally isomorphic to an affine variety over k.

Example: Consider $(X, \mathcal{T}_X, \mathcal{O}_X)$ where $X = \mathbb{A}^2 \setminus (0,0)$, $\mathcal{T}_X$ is the subspace topology on $\mathcal{T}_{\mathbb{A}^2}$ and $\mathcal{O}_X$ is the sheaf induced by $\mathcal{O}_{\mathbb{A}^2}$. We can see that this is an example of a variety since for any $x \in X$ we can choose any open neighbourhood of x that doesn't contain the origin and it will be isomorphic to the same open set in $\mathbb{A}^2$, which is an affine variety. I believe it can be proven that X is not an affine variety but I will not do so here.

Very fluffy. Proof machine proof?

What exactly does this mean?

Really, Why?

(h) affine scheme

An **affine scheme** is an element of the image of Spec, defined in part (q).

Example: We will consider Spec($\mathbb{Z}$) and have an in depth look at the topology and structure sheaf. In particular we show the following:

GOOD!

To show:

(a) $\text{Spec}(Z) = \{p\mathbb{Z} \mid p = 0 \text{ or } p \text{ prime}\}$.

(b) Closed sets are of the form $V(n\mathbb{Z}) = \{p\mathbb{Z} \in \text{Spec}(\mathbb{Z}) \mid n \text{ is a multiple of } p\}$.

(c) For $n \in \mathbb{Z}$ the basic set is $X_n = \{p\mathbb{Z} \mid p \text{ is not a prime divisor of } n\}$ if $n \neq 0$ and $X_0 = \emptyset$. Furthermore, every open set is a basic set.

(a) Since $\mathbb{Z}$ is an integral domain, (0) is a prime ideal. Also, $\mathbb{Z}$ is a principal ideal domain, so every ideal is of the form $n\mathbb{Z}$ for $n \in \mathbb{Z}$.

To show: If $p\mathbb{Z}$ is a prime ideal then p is prime.

To show: If p is not prime then $p\mathbb{Z}$ is not a prime ideal.

Assume p is not prime.

To show: $p\mathbb{Z}$ is not a prime ideal.

To show: There exists $x, y \in \mathbb{Z}$ such that $xy \in p\mathbb{Z}$ but $x, y \notin p\mathbb{Z}$.

Let $p = xy$ where neither x or y are 1 or p. Then $xy = p \in pZ$ but neither x or y are multiples of pZ and so aren't in $p\mathbb{Z}$.

So $p\mathbb{Z}$ is not a prime ideal.

So $\text{Spec}(\mathbb{Z}) = \{p\mathbb{Z} \mid p = 0 \text{ or } p \text{ prime}\}$.

(b) By definition of the Zariski topology, and using the fact that $\mathbb{Z}$ is a PID, closed sets are of the form $V(n\mathbb{Z}) = \{p\mathbb{Z} \in \text{Spec}(\mathbb{Z}) \mid n\mathbb{Z} \subseteq p\mathbb{Z}\}$. The result comes from the fact that $n\mathbb{Z} \subseteq p\mathbb{Z}$ if and only if n is a multiple of p.

? so these are? What is a good characterization of closed sets?

(c) By definition of the basic set $X_n = \{p \in \text{Spec}(\mathbb{Z}) \mid n \neq 0 \text{ in } \mathbb{Z}/p\mathbb{Z}\}$. An integer is zero in $\mathbb{Z}/p\mathbb{Z}$ if and only if that integer is a multiple of p. In other words $p\mathbb{Z} \in X_n$ if and only if it is not a prime divisor of n. In particular, $X_0 = \emptyset$ and $X_1 = \text{Spec}(\mathbb{Z})$. The prime ideal (0) is in every open set except the empty set. We can also now see that $V(n\mathbb{Z}) = \text{Spec}(\mathbb{Z}) \setminus X_n$, and since every closed set is of the form $V(n\mathbb{Z})$, this means that every open set is a basic set.

Say this better with more punch/impact. format would help.

Now to look into the structure sheaf: by definition,

$$\mathcal{O}_X(X_n) = \mathbb{Z}\left[\frac{1}{n}\right] = \left\{\frac{m}{n^k} \mid m \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}\right\}$$

a ring I hope

which is the subset of $\mathbb{Q}$ of fractions which have a denominator equal to a power of n. We can note that since m divides n if and only if every divisor of m is a divisor of n, we have:

$$X_n \subseteq X_m \text{ if and only if } m \text{ divides } n.$$

So we can show that if $X_n \subset X_m$ then we can write $n = sm$ for some $s \in \mathbb{Z}$ and thus the restriction map is given by:

format better

Say with more punch/impact.

$$\text{res}^{X_m}_{X_n} : \mathbb{Z}\left[\frac{1}{m}\right] \to \mathbb{Z}\left[\frac{1}{n}\right], \qquad \frac{x}{m^k} \mapsto \frac{xs^k}{n^k}.$$

We may note that, viewed as a map from $\mathbb{Q}$ to $\mathbb{Q}$, this is just the identity, since $s = n/m$ implies

healthy

probably good to format in theorem/proof format.

$$\frac{xs^k}{n^k} = \frac{x}{n^k}\frac{n^k}{m^k} = \frac{x}{m^k}.$$

(i) scheme

A **scheme** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme.

Example: [Har, §6.2, Example 5] Consider the affine scheme $(X, \mathcal{T}, \mathcal{O}_X)$ where $X = \operatorname{Spec} \mathbb{C}[f,g]$. Define $(\tilde{X}, \mathcal{T}_{\tilde{X}}, \mathcal{O}_{\tilde{X}})$ as follows:

- $\tilde{X} = X \setminus (f,g)$, where (f,g) is the ideal generated by f and g.
- $\mathcal{T}_{\tilde{X}} =$ the subspace topology.
- $\mathcal{O}_{\tilde{X}}$ the sheaf defined by $\mathcal{O}_{\tilde{X}}(U) = \mathcal{O}_X(U)$.

To show: $(\tilde{X}, \mathcal{T}_{\tilde{X}}, \mathcal{O}_{\tilde{X}})$ is a scheme.

To show: $(\tilde{X}, \mathcal{T}_{\tilde{X}}, \mathcal{O}_{\tilde{X}})$ is locally isomorphic to an affine scheme.

To show: $(\tilde{X}, \mathcal{T}_{\tilde{X}}, \mathcal{O}_{\tilde{X}})$ is locally isomorphic to $(X, \mathcal{T}, \mathcal{O}_X)$.

OK **To show:** If $p \in \tilde{X}$ then there exists an open set $U \in \mathcal{T}_{\tilde{X}}$ with $p \in U$, an open set $V \in \mathcal{T}$ and an isomorphism $f : (U, \mathcal{T}_U, \mathcal{O}_U) \to (V, \mathcal{T}_V, \mathcal{O}_V)$.

Assume $p \in \tilde{X}$.

[illegible]

We can write $\tilde{X} = X_f \cup X_g$ where X_f and X_g are the basic sets as defined in the definition of Spec. As a reminder, $X_f = \{p \in \operatorname{Spec} \mathbb{C}[f,g] \mid f \notin p\}$ and X_g is defined in the same way. Recall that X_f and X_g are open in $\operatorname{Spec} \mathbb{C}[f,g]$.

If $p \in X_f$ then define $U = V = X_f$, otherwise $p \in X_g$ and we can define $U = V = X_g$. In either case, we let f be the identity.

Kind of [illegible]. I can't really parse.

So $(\tilde{X}, \mathcal{T}_{\tilde{X}}, \mathcal{O}_{\tilde{X}})$ is locally isomorphic to an affine scheme.

So $(\tilde{X}, \mathcal{T}_{\tilde{X}}, \mathcal{O}_{\tilde{X}})$ is a scheme.

(j) (topological) manifold

The **affine topological manifold** is the ringed space $(\mathbb{R}^n, \mathcal{T}^{std}, C^0)$.

A **topological manifold** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine topological manifold.

Example: Consider the ringed space $(S^n, \mathcal{T}_{S^n}, \mathcal{O}_{S^n})$ where S^n is the unit sphere in $\mathbb{R}^{n+1}$, $\mathcal{T}_{S^n}$ is the subspace topology on $S^n \subseteq \mathbb{R}^{n+1}$ and $\mathcal{O}_{S^n}$ is the sheaf given by $\mathcal{O}_{S^n}(U) = C^0(U)$ for $U \in \mathcal{T}_{S^n}$.

To show: $(S^n, \mathcal{T}_{S^n}, \mathcal{O}_{S^n})$ is a topological manifold.

To show: $(S^n, \mathcal{T}_{S^n}, \mathcal{O}_{S^n})$ is locally isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{T}, C^0)$.

To show: If $v \in S^n$ then there exists $U \in \mathcal{T}_{S^n}$ with $v \in U$ and $V \in \mathcal{T}^{std}$ and an isomorphism of ringed spaces $(U, \mathcal{T}_U, \mathcal{O}_U) \to (V, \mathcal{T}_V, \mathcal{O}_V)$.

Assume $v \in S^n$. We may assume without loss of generality that $v = (0,0,...,0,1)$, since otherwise we simply apply a ringed space isomorphism that takes v to the point $(0,0,...,0,1)$.

To show: There exists $U \in \mathcal{T}_{S^n}$ with $v \in U$ and $V \in \mathcal{T}^{std}$ and an isomorphism $f : (U, \mathcal{T}_U, \mathcal{O}_U) \to (V, \mathcal{T}_V, \mathcal{O}_V)$.

Let $U = \{(v_0, ..., v_n) \in S^n \mid v_n > 0\}$, in words the open hemisphere containing v bounded by the hyper-plane through the origin orthogonal to v.

Let $V = \{(v_1, ..., v_n) \in \mathbb{R}^n \mid v_1^2 + ... + v_n^2 < 1\}$, in words the open unit ball at the origin.

Define $f : (U, \mathcal{T}_U) \to (V, \mathcal{T}_V)$ by $f(v_0, ..., v_n) = (v_0, ..., v_{n-1})$. Define a family of maps $\{h_S : \mathcal{O}_U(S) \to \mathcal{O}_V(f(S))\}_{S \in \mathcal{T}_U}$ by $h_S(\varphi) = \varphi \circ f^{-1}$. This is well defined since f is a homeomorphism and the composition of continuous maps are continuous.

If f is an isomorphism and each h_S is a ring isomorphism then they define an isomorphism of ringed spaces as required.

To show:

(a) f is a homeomorphism.

(b) If $S \in \mathcal{T}_U$ then h_S is a ring isomorphism.

(a) f is just a projection and it's fairly straightforward to prove it's a homeomorphism. If I have time I'll come back to this.

(b) Assume $S \in \mathcal{T}_U$.

To show: h_S is a ring isomorphism.

Note that $\mathcal{O}_U = C^0(U)$.

To show:

- (ba) If $\phi, \psi \in C^0(U)$ then $h_S(\phi + \psi) = h_S(\phi) + h_S(\psi)$ and $h_S(\phi\psi) = h_S(\phi)h_S(\psi)$
- (bb) There exists an inverse function $h_S^{-1} : \mathcal{O}_V(f(S)) \to \mathcal{O}_U(S)$.

(ba) Assume $\phi, \psi \in C^0(U)$.

To show: $h_S(\phi + \psi) = h_S(\phi) + h_S(\psi)$ and $h_S(\phi\psi) = h_S(\phi)h_S(\psi)$.

$$h_S(\phi + \psi) = (\phi + \psi) \circ f^{-1} = \phi \circ f^{-1} + \psi \circ f^{-1} = h_S(\phi) + h_S(\psi),$$
$$h_S(\phi\psi) = (\phi\psi) \circ f^{-1} = (\phi \circ f^{-1})(\psi \circ f^{-1}) = h_S(\phi)h_S(\psi).$$

(bb) Define $h_S^{-1} : \mathcal{O}_V(f(S)) \to \mathcal{O}_U(S)$ by $h_S^{-1}(\phi) = \psi \circ f$.

$$h_S \circ h_S^{-1}(\phi) = h_S(\phi \circ f) = \psi \circ f \circ f^{-1} = \phi,$$
$$h_S^{-1} \circ h_S(\phi) = h_S^{-1}(\phi \circ f^{-1}) = \phi \circ f^{-1} \circ f = \phi.$$

So h_S is a ring isomorphism.

So we have an isomorphism of ringed spaces.

So $(S^n, \mathcal{T}_{S^n}, \mathcal{O}_{S^n})$ is locally isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{T}, \mathcal{C}^0)$.

So $(S^n, \mathcal{T}_{S^n}, \mathcal{O}_{S^n})$ is a topological manifold.

(k) smooth manifold

The **affine smooth manifold** is the ringed space $(\mathbb{R}^n, \mathcal{T}^{std}, \mathcal{C}^\infty)$.

A smooth manifold is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine smooth manifold.

Example: If we consider the last example, $(S^n, \mathcal{T}_{S^n}, \mathcal{O}_{S^n})$, with $\mathcal{O}_{S^n}(U) = \mathcal{C}^\infty(U)$ instead of $\mathcal{C}^0(U)$ then we have a smooth manifold. The proof is identical except we replace $\mathcal{C}^0$ with $\mathcal{C}^\infty$ everywhere and view the relevant maps as smooth instead of simply continuous.

(l) $\mathcal{C}^r$-manifold

The **affine $\mathcal{C}^r$-manifold** is the ringed space $(\mathbb{R}^n, \mathcal{T}^{std}, \mathcal{C}^r)$.

A **$\mathcal{C}^r$-manifold** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine $\mathcal{C}^r$-manifold.

Example: By noting that any smooth map is also $\mathcal{C}^r$ for any $r \in \mathbb{Z}_{>0}$, $(S^n, \mathcal{T}_{S^n}, \mathcal{C}^r)$ is an example of a $\mathcal{C}^r$-manifold, following the proof of part (j).

But there are C^r maps that are not C^∞! So this still needs careful checking.

(m) complex manifold

The **affine complex manifold** is the ringed space $(\mathbb{C}^n, \mathcal{T}^{std}, \mathcal{C})$, where $\mathcal{C}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$.

A **complex manifold** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine complex manifold.

Example: [Oec] The Riemann sphere is a complex manifold. Let $X = \mathbb{C} \cup \{\infty\}$. Define a topology $\mathcal{T}_X$ on X as follows: $U \in \mathcal{T}_X$ if and only if $U \in \mathcal{T}^{std}$ of $U = V \cup \{\infty\}$ where $V \subseteq \mathbb{C}$ and $\mathbb{C} \setminus V$ is compact in $\mathbb{C}$. Define a structure sheaf on X by

Why did you choose this reference and not Harder or not as done in class

$$\mathcal{O}_X(U) = \begin{cases} \mathcal{C}(U), & U \in \mathcal{T}^{std}, \\ \mathcal{C}(V), & U = V \cup \{\infty\} \in \mathcal{T}_X. \end{cases}$$

To show: $(X, \mathcal{T}_X, \mathcal{O}_X)$ is a complex manifold.

To show: $(X, \mathcal{T}_X, \mathcal{O}_X)$ is locally isomorphic to $(\mathbb{C}, \mathcal{T}^{std}, \mathcal{C})$.

How does this relate to Harder and/or what is done in class?

To show: If $p \in X$ then there exists $U \in \mathcal{T}_X$, $V \in \mathcal{T}^{std}$ and an isomorphism $f : (U, \mathcal{T}_U, \mathcal{O}_U) \to (V, \mathcal{T}_V, \mathcal{O}_V)$.

Assume $p \in X$.

If $p \neq \infty$ then we can define $U = \mathbb{C} \in \mathcal{T}$ and $V = \mathbb{C} \in \mathcal{T}^{std}$ and so $(U, \mathcal{T}_U, \mathcal{O}_U)$ and $(V, \mathcal{T}_V, \mathcal{O}_V)$ are the same space, hence isomorphic.

Now assume $p = \infty$. Let $U = X \setminus \{0\}$ and $V = \mathbb{C}$, so that $(V, \mathcal{T}_V, \mathcal{O}_V) = (\mathbb{C}, \mathcal{T}^{std}, \mathcal{C})$.

To show: There exists an isomorphism $f : (U, \mathcal{T}_U, \mathcal{O}_U) \to (\mathbb{C}, \mathcal{T}^{std}, \mathcal{C})$.

Let $f : (U, \mathcal{T}_U) \to (\mathbb{C}, \mathcal{T}^{std})$ be defined by

$$f(z) = \begin{cases} \frac{1}{z}, & z \in \mathbb{C}\setminus\{0\}, \\ 0, & z = \infty. \end{cases}$$

For $S \in \mathcal{T}_U$, define $h_S : \mathcal{O}_U(S) \to \mathcal{C}(f(S))$ by $h_S(\varphi) = \varphi \circ f^{-1}$. This is a well-defined function since f is holomorphic (where it is defined). The proof of this is omitted.

To show:

(a) f is a homeomorphism.

(b) If $S \in \mathcal{T}_U$ then h_S is a ring isomorphism.

(a) To show:

- (aa) f is continuous.
- (ab) There exists a continuous map $g : (\mathbb{C}, \mathcal{T}^{std}) \to (U, \mathcal{T}_U)$ such that $f \circ g = \text{id}$ and $f \circ g = \text{id}$.

(aa) **To show:** If $V \in \mathcal{T}^{std}$ then $f^{-1}(V) \in \mathcal{T}_U$.

Assume $V \in \mathcal{T}^{std}$.

To show: $f^{-1}(V) \in \mathcal{T}_U$.

We know that f is continuous on $\mathbb{C}\setminus\{0\}$ so if V does not contain 0 then $f^{-1}(V)$ is open.

Now we only need to consider open neighbourhoods 0. Since open balls are a basis for the standard topology, it is sufficient to assume $V = B_r$, a ball of radius r centered at the origin. In this case,

$$f^{-1}(B_r) = \{z \in U \mid |z| > \frac{1}{r}\} \cup \{\infty\}$$

which is open in U because

$$\mathbb{C}\setminus\{z \in U \mid |z| > \frac{1}{r}\} = \{z \in \mathbb{C} \mid z \leq 1/r\}$$

which is a compact set in $\mathbb{C}$, being closed and bounded.

Thus f is a continuous function.

(ab) Let $g : (\mathbb{C}, \mathcal{T}^{std}) \to (U, \mathcal{T}_U)$ be defined by

$$g(z) = \begin{cases} \frac{1}{z} & z \neq 0 \\ \infty & z = 0. \end{cases}$$

To show:

(aba) g is continuous.

(abb) $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$.

(aba) **To show:** If $V \in \mathcal{T}$ then $g^{-1}(U) \in \mathcal{T}^{std}$.

Assume $V \in \mathcal{T}$.

To show: $g^{-1}(V) \in \mathcal{T}^{std}$.

Again, we know that g is continuous on $\mathbb{C} \setminus \{0\}$ so if V does not contain ∞ then $g^{-1}(V)$ is open in $\mathbb{C}$.

Now we only need to consider open sets about ∞. Since the compact sets in $\mathbb{C}$ are those that are closed and bounded, one can check that a neighbourhood basis of ∞ in $\mathcal{T}_U$ are given by sets of the form $\{z \in U \mid |z| > r\} \cup \{\infty\}$. Thus by noting that

$$g^{-1}(\{z \in U \mid |z| > r\} \cup \{\infty\}) = \{z \in \mathbb{C} \mid |z| < \frac{1}{r}\}$$

is open in $\mathbb{C}$, we can see that $f^{-1}(V)$ is open in $\mathbb{C}$.

(abb)

$$g \circ f(z) = \begin{cases} g(1/z) & z \in \mathbb{C} \setminus \{0\} \\ g(0) & z = \infty \end{cases} = z$$

$$f \circ g(z) = \begin{cases} f(1/z) & z \neq 0 \\ f(\infty) & z = 0 \end{cases} = z$$

So f is a homeomorphism

(b) The proof of this part is practically identical to part (b) of the proof done in the (topological) manifold section. It can be copied here verbatim, since holomorphic functions work the same way as continuous ones.

(n) projective space

[Har, §8.1.1] gives another definition of projective space. Go there for the details I overlook.

Let A be a commutative ring with identity and let $S = \operatorname{Spec} A$. For $i = 0, ..., n$ let

$$U_i = \operatorname{Spec}(A[T_{i,0}, ..., T_{i,n}]/(T_{i,i} - 1)).$$

We can show that each U_i is a copy of $\mathbb{A}^n_S$. If we write $t_{i,j}$ for the images of $T_{i,j}$ in the quotient $A[T_{i,0},...,T_{i,n}]/(T_{i,i}-1)$ then we can write $U_i = \mathrm{Spec}(A[t_{i,0},t_{i,1},...,t_{i,i-1},t_{i,i+1},..,t_{i,n}])$ since $t_{i,i}=1$. For any index j we define the following open subscheme of U_i:

$$U_{i,j} = \mathrm{Spec}(A[T_{i,0},...,T_{i,n}]/(T_{i,i}-1))_{T_{i,j}}$$
$$= \mathrm{Spec}(A[t_{i,0},t_{i,1},...,t_{i,i-1},t_{i,i+1},..,t_{i,n},t_{i,j}^{-1}]).$$

Here I believe the subscript $T_{i,j}$ denotes the localisation, which leads to the $t_{i,j}^{-1}$ in the next line. We have an isomorphism

$$f_{i,j} : U_{i,j} \to U_{j,i}$$

which on the level of rings is given by

$$\phi_{i,j} : A[t_{i,0},t_{i,1},...,t_{i,i-1},t_{i,i+1},..,t_{i,n},t_{i,j}^{-1}] \to A[t_{i,0},t_{i,1},...,t_{i,i-1},t_{i,i+1},..,t_{i,n},t_{j,i}^{-1}]$$
$$\phi_{i,j}(t_{j,\nu}) \mapsto t_{i,\nu} \cdot t_{i,j}^{-1}$$

Thus we define

$$\mathbb{P}^n_A = \left(\coprod_{i=0,...,n} U_i \right) / \sim$$

where $\sim$ is the equivalence relation defined by $u_i \sim u_j$ if and only if $u_i \in U_j$, $u_j \in U_{j,i}$ and $\phi_{i,j}(u_i) = u_j$.

how do you put an element of u_i into ϕ_{ij}?

We give $\mathbb{P}^n_A$ the quotient topology, where each U_i has the Zariski topology of affine space.

We define a sheaf $\mathcal{O}_{\mathbb{P}^n_A}$ on $\mathbb{P}^n_A$ by having

$$\mathcal{O}_{\mathbb{P}^n_A}(U_i) = A[\widetilde{T_{i,0},...,}T_{i,n}]/(T_{i,i}-1)$$

and using the $\phi_{i,j}$ to glue $\mathcal{O}(U_i)\mid_{U_i\cap U_j}$ with $\mathcal{O}(U_j)\mid_{U_i\cap U_j}$.

(I couldn't find what Harder meant by the wide tilde - my first guess would be homogeneous polynomials but I don't think that's it.)

(I would like understand this construction better - particularly the motivation and how it connects the other definition of projective space.)

(o) CW space

[Hat, p.519] A **CW complex** is a topological space $(X,\mathcal{T})$ that can be constructed in the following way:

1. Start with a discrete set X^0 and the discrete topology $\mathcal{T}_0$ on X^0.
2. Inductively form the space X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$. In other words X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$. The topology $\mathcal{T}_n$ on X^n is the quotient topology induced from $\mathcal{T}_{n-1}$.

is this parsable by a human?

3. Either let $X = X^n$ for some $n \in \mathbb{Z}_{\geq 0}$ or let $X = \bigcup_{n \in \mathbb{Z}_{\geq 0}} X^n$. In the latter case, $\mathcal{T}$ is the weak topology: A set $A \subseteq X$ is open if and only if $A \cap X^n$ is open in X^n for each n.

Example: We consider $S^n = D^n/\partial D^n$ with the quotient topology induced by the subset topology on $D^n \subset \mathbb{R}^n$, where $\mathbb{R}^n$ has the standard topology.

To show: S^n can be given the structure of a CW complex.

To show: There exist a sequence $X^0, ..., X^n$ of topological spaces such that $S^n = X^n$ and each X^k is constructed as per the definition of a CW complex.

Let $X^{n-1} = X^{n-2} = ... = X^0 = \{e_0\}$.

Let $X^n = X^0 \sqcup_\phi D^n$ where $\phi : S^{n-1} \to X^0$ is the constant map $\phi(v) = e_0$.

To show: $S^n = X^n$.

X^n is an n-disk with the boundary identified to a point, e_0. Thus $X^n = S^n$.

You need to specify a homeomorphism, check that it is continuous and check that its inverse is continuous

(p) simplicial set

[Sno10] Let Δ be the category whose objects are finite sets of the form $\{0, 1, 2, ..., n\}$ and whose morphims are order preserving functions.

A **simplicial set** is a contravariant functor $\Delta \to$ Set.

Example: Consider the functor $F : \Delta \to$ Set where

and what space is this??

$$F(\{0, 1, ..., n\}) = \{\text{functions } \{0, 1, ..., n\} \to \mathbb{R}\}$$

and if $f : \{0, 1, ..., n\} \to \{0, 1, ..., m\}$ is an arrow in Δ then $F(f) : F(\{0, 1, ..., m\}) \to F(\{0, 1, ..., n\})$ is defined by $F(f)(\varphi) = \varphi \circ f$. This is a functor since

$$F(f \circ g)(\varphi) = (f \circ g) \circ \varphi = f \circ (g \circ \varphi) = F(f) \circ F(g)(\varphi).$$

(q) spectrum

Spec is the contravariant functor

$$\text{Spec} : \{\text{commutative rings}\} \to \{\text{ringed spaces}\}$$

where

- $X = \text{Spec}(R)$ is the set of prime ideals of R.
- The topology on Spec(R) is the Zariski topology i.e. the topology with closed sets

$$V(I) = \{p \in \text{Spec}(R) \mid I \subseteq p\}$$

where I is an ideal of R. The basis sets

$$X_g = \{p \in X \mid g \neq 0 \text{ in } A/p\}$$

form a basis of open sets for the topology.

- The structure sheaf $\mathcal{O}_X$ on X is determined by

$$\mathcal{O}_X(X_g) = R\left[\frac{1}{g}\right]$$

where $R[1/g] = \{f/g^k \mid f \in R, k \in \mathbb{Z}_{\geq 0}\}$. If $X_h \subset X_g$ then the restriction map is given by

$$\operatorname{res}^{X_g}_{X_h} : R\left[\frac{1}{h}\right] \to R\left[\frac{1}{g}\right], \quad \frac{f}{h^m} \to \frac{fs^m}{g^{mn}}$$

with $g^n = sh$ where $s \in R$ and $n \in \mathbb{Z}_{>0}$.

Example: Affine space can be defined as the spectrum of a polynomial ring. As an example we will look at $S = \operatorname{Spec} \mathbb{C}[X]$. We look at the set S, the topology $\mathcal{T}_S$ and the sheaf $\mathcal{O}_S$, each as defined in the definition of Spec. In particular:

To show:

(a) S is in bijection to $\mathbb{C}$.

(b) Under this bijection, the topology $\mathcal{T}_S$ is equivalent the topology $\mathcal{T}$ defined in the definition of affine space given in (d), i.e. $\mathcal{T}$ has closed sets

$$V'(P) = \{x \in \mathbb{C} \mid \text{ if } f \in P \text{ then } f(x) = 0\}.$$

(a) **To show:** There is a one-to-one correspondence between elements of S and $\mathbb{C}$ i.e. $S = \mathbb{C}$.

To show: Prime ideals of $\mathbb{C}[X]$ are of the form $(X - z)$ for $z \in \mathbb{C}$. isn't 0 prime?

Recall that $\mathbb{C}[X]$ is a principal ideal domain, hence the proper ideals of $\mathbb{C}[X]$ are of the form (f) where $f \in \mathbb{C}[X]$ is a monic polynomial.

To show: If (f) is a prime ideal of $\mathbb{C}[X]$ then f is a monic polynomial of degree 1.

To show: If f is not a monic polynomial of degree 1 then (f) is not prime.

Assume f is not a monic polynomial of degree 1. Then there exist non-constant $g, h \in \mathbb{C}[X]$ such that $f = gh$. Neither g or h can be elements of (f) and since $gh \in (f)$ this shows that (f) is not prime.

So every prime ideal in $\mathbb{C}[X]$ is of the form $(X - z)$ for $z \in \mathbb{C}$.

So S is in bijection with $\mathbb{C}$ by the map $(X - z) \leftrightarrow z$.

(b) We need to show that every closed set in $\mathcal{T}_S$ is closed in $\mathcal{T}$ and vice versa. It is sufficient to show that if $I \subset \mathbb{C}[X]$ then $V(I) = V'(I)$.

Assume $I \subset \mathbb{C}[X]$.

To show: $V(I) = V'(I)$.

To show: If $z \in \mathbb{C}$ then the following statements are equivalent:

(i) $I \subset (X - z)$.

(ii) If $f \in I$ then $f(z) = 0$.

Indeed this is true by noting that both statements are equivalent to the statement: If $f \in I$ then $f(X)$ has a factor $X - z$.

So the two topologies are equivalent.

Both (a) and (b) together imply that in this case $(\mathbb{C}, \mathcal{T})$ and $(\mathrm{Spec}\,\mathbb{C}[X], \mathcal{T}_S)$ are equivalent as topological spaces. This does not mean the two definitions of affine space are equivalent, since as far as I can tell the sheaf of regular functions is different to the sheaf defined in Spec.

(r) orbifold

chapter/verse?

good reference

[Sno11] An n-dimensional **orbifold chart** on a topological space X is a 3-tuple $(\tilde{U}, G, \pi)$ where:

- $\tilde{U}$ is open in $\mathbb{R}^n$.
- G is a finite group of homeomorphisms of $\tilde{U}$.
- $\pi : \tilde{U} \to X$ is a map defined by $\pi = \bar{\pi} \circ p$ where $p : \tilde{U} \to \tilde{U}/G$ is the orbit map and $\bar{\pi} : \tilde{U}/G \to X$ is a map that induces a homeomorphism of $\tilde{U}/G$ onto an open subset U of X.

An **embedding** $\lambda : (\tilde{U}_1, G_1, \pi_1) \to (\tilde{U}_2, G_2, \pi_2)$ is a smooth embedding $\lambda : \tilde{U}_1 \to \tilde{U}_2$ such that $\pi_2 \circ \lambda = \pi_1$.

Let $(\tilde{U}_1, G_1, \pi_1)$ and $(\tilde{U}_2, G_2, \pi_2)$ be two orbifold charts on a topological space X and let $U_i = \pi_i(\tilde{U}_i)$ for $i = 1, 2$ and $x \in U_1 \cap U_2$. The two charts are **compatible** if there exists an open neighbourhood $V \subseteq U_1 \cap U_2$ of x and a chart $(\tilde{V}, H, \phi)$ such that $\phi(\tilde{V}) = V$ and there are two embeddings $\lambda_i : (\tilde{V}, H, \phi) \to (\tilde{U}_i, G_i, \pi_i)$.

An n-dimensional **orbifold atlas** on X is a collection $\mathcal{U} = \{(\tilde{U}_\alpha, G_\alpha, \pi_\alpha)\}_{\alpha \in A}$ of compatible n-dimensional orbifold charts which cover X.

An n-dimensional **orbifold** is a paracompact Hausdorff space X together with an n-dimensional orbifold atlas of charts.

Example Let $X = \mathbb{R}^2/(\mathbb{Z}/2)$ where $\mathbb{Z}/2$ acts on $\mathbb{R}^2$ by $(x, y) \mapsto (x, -y)$.

Define $\tilde{U} = \mathbb{R}^2$, $G = \mathbb{Z}/2$ and $p : \tilde{U} \to X$ the orbit map.

We show that X together with the (to be shown) orbifold atlas $\{(\tilde{U}, G, p)\}$ is an orbifold.

To show:

(a) X is paracompact and Hausdorff.

(b) $\{(\tilde{U}, G, p)\}$ is an orbifold atlas on X.

(a) **To show:**

- (aa) X is paracompact.
- (ab) X is Hausdorff.

(aa) I will do this part if I have time - I'm leaving it for now because it's not strictly algebraic geometry.

good reasoning - hard to argue with this.

(ab) **To show:** If $[x], [y] \in X$ then there exists open sets U and V of X such that $[x] \in U$, $[y] \in V$ and $U \cap V = \emptyset$.

Assume $[x], [y] \in X$.

To show: There exists open sets U and V of X such that $[x] \in U$, $[y] \in V$ and $U \cap V = \emptyset$.

Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Note that $[x] = [(x_1, x_2)] = [(x_1, -x_2)]$ and $[y] = [(y_1, y_2)] = [(y_1, -y_2)]$. Let d be the minimum distance between the points (x_1, x_2), $(x_1, -x_2)$, (y_1, y_2) and $(y_1, -y_2)$, using the standard metric on $\mathbb{R}^2$.

Define $\varepsilon = d/2$ and let $U = p(B_\varepsilon(x))$ and $V = p(B_\varepsilon(y))$, where $B_r(v)$ is the open ball of radius r centered at v.

To show:

- (aaa) $[x] \in U$ and $[y] \in V$.
- (aab) U and V are open in X.
- (aac) $U \cap V = \emptyset$.

(aaa) Since $x \in B_\varepsilon(x)$ and $y \in B_\varepsilon(y)$ we conclude that $[x] = p(x) \in U$ and $[y] = p(y) \in V$.

(aab) We can check that $p^{-1}(U) = B_\varepsilon((x_1, x_2)) \cup B_\varepsilon((x_1, -x_2))$ and is therefore open in $\mathbb{R}^2$ as the union of two open balls. Thus by the definition of the quotient topology, U is open in X. The same is true for V by an identical argument.

(aac) **To show:** If $[u] \in U$ then $[u] \notin V$.

Assume $u = (u_1, u_2) \in \mathbb{R}^2$ such that $[u] \in U$.

To show: $[u] \notin V$.

Since $p^{-1})(U) = B_\varepsilon((x_1, x_2)) \cup B_\varepsilon((x_1, -x_2))$ we know that u is within ε units of x or $(x_1, -x_2)$. By our choice of ε we can conclude that u cannot be within ε units of y or $(y_1, -y_2)$ and hence $[u]$ cannot be an element of V.

So X is Hausdorff.

So X is paracompact and Hausdorff.

(b) **To show:** $(\tilde{U}, G, p)$ is an orbifold chart on X.

To show:

(a) $\tilde{U}$ is open in $\mathbb{R}^2$.

(b) G is a group of homeomorphisms on $\mathbb{R}^2$.

(c) $p = \bar{\pi} \circ p$ where $\bar{\pi} : \tilde{U}/G \to X$ induces a homeomorphism from $\tilde{U}/G$ to an open set U of X.

(a) $\tilde{U} = \mathbb{R}^2$ is open in $\mathbb{R}^2$ by the definition of a topological space.

(b) G is a group of homemorphisms on $\mathbb{R}^2$ by the defined group action.

(c) Noting that $\tilde{U}/G = \mathbb{R}^2/G = X$, we simply define X to be the identity. Since the identity is a homeomorphism on X, this condition holds.

So $(\tilde{U}, G, p)$ is an orbifold chart on X

So $\{(\tilde{U}, G, p)\}$ is an orbifold atlas on X

So X together with $\{(\tilde{U}, G, p)\}$ is an orbifold.

(s) algebraic stack

The following definition is from [Beh14, Definition 1.148]. It would require a bit of unpacking for me to work with. good reference.

An **algebraic stack** is a group fibration $\mathfrak{X}$ over $\mathscr{S}$ such that

(i) $\mathscr{X}$ admits a versal family x/Γ_0 whose symmetry groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ is an algebraic groupoid.

(ii) the tautological morphism of fibrations

$$\mathfrak{X} \to (\Gamma - \text{torsors})$$

$$y \mapsto \underline{\mathrm{Isom}}(y, x)$$

is an equivalence.

good paragraph

The current idea I have in my head is that fine moduli spaces can actually be viewed as spaces but coarse moduli spaces must be viewed as stacks. In class we looked at $\mathbb{R}/(\mathbb{Z}/2\mathbb{Z})$ as the coarse moduli space of circles in $\mathbb{R}^2$ centered at the origin. This is a stack that somehow keeps track of the stabilisers, in this case the point 0 has two stabilisers but the other points just have 1. So my current picture of a stack is the positive number line with two points at 0. I hope to get a better idea of stacks by going to the number theory seminar.

(t) curve

[Har77, p.4] Let k be a field and let $f \in k[x,y]$ be an irreducible polynomial. A **curve** C is a set

Only a set?

$$C = \{(x,y) \in k^2 \mid f(x,y) = 0\}.$$

Let

Example: Let $k = \mathbb{C}$. ~~Consider the polynomial~~ $\cancel{f(x,y) = y^2 - x^3 - ax - b}$ ~~where~~ $a, b \in \mathbb{C}$. ~~Then~~ An elliptic curve over $\mathbb{C}$ is

$$C = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\} = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 + ax + b\}$$

~~is called an elliptic curve over $\mathbb{C}$.~~

(u) surface

[Har77, p.4] Let k be a field and let $f \in k[x,y,z]$ be an irreducible polynomial. A **surface** S is a set

$$S = \{(x,y,z) \in k^3 \mid f(x,y,z) = 0\}.$$

Example: Let $k = \mathbb{R}$ and let

$$f(x,y,z) = (x^2+y^2+z^2+2y-1)[(x^2+y^2+z^2-2y-1)^2-8z^2]+16xz(x^2+y^2+z^2-2y-1).$$

Then the surface

$$K = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$$

is an immersion of the klein bottle in $\mathbb{R}^3$ (See `http://mathworld.wolfram.com/KleinBottle.html`).

means?

good reference

(v) hypersurface

[Har77, p.4] Let k be a field and let $n \in \mathbb{Z}_{>3}$ and $f \in k[x_1, x_2, ..., x_n]$ be an irreducible polynomial. A **hypersurface** H is a set

$$H = \{(x_1, x_2, ..., x_n) \in k^n \mid f(x_1, x_2, ..., x_n) = 0\}.$$

Example: Let $k = \mathbb{R}$ and define $f(x_1, ..., x_n) = x_1^2 + x_2^2 + ... + x_n^2 - 1$. Then

$$S^{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid f(x_1, ..., x_n) = 0\} = \{(x_1, ..., x_n) \in \mathbb{R}^{n+1} \mid x_1^2 + ... + x_n^2 = 1\}$$

is a hypersurface for $n > 3$.

(w) moduli space

Loosely speaking a **moduli space** is space whose points represent objects in some family. The geometric structure of the space should in a sense provide information on the family of objects it represents.

For example, we can view $\mathbb{R}P^1$ as the moduli space of lines through the origin in $\mathbb{R}^2$.

(x) perfectoid space

chapter/verse?

Definitions from [Sch12].

Let K be a perfectoid field.

An **affinoid K-algebra** is a pair (R, R^+) consisting of a Tate k-algebra and an open integrally closed subring $R^+ \subset R^0$, where R^0 denotes the set of powerbounded elements.

A **perfectoid affinoid K-algebra** is an affinoid K-algebra (R, R^+) such that R is a perfectoid K-algebra.

Given a perfectoid affinoid K-algebra (R, R^+) we have associated an affinoid adic space $X - \mathrm{Spa}(R, R^+)$. We call these spaces **affinoid perfectoid spaces.** (The method of forming this association is outlined in section 6 of Scholze's paper)

A **perfectoid** space is an adic space over K that is locally isomorphic to an affinoid perfectoid space.

definition of this?

(y) coarse moduli space

[Beh14, Definition 1.18] A **coarse moduli space** is a space M such that the first two conditions of the definition in (z) are satisfied, and moreover M carries the finest topology that makes (ii) true.

Example: In class we talked about $\mathbb{R}/(\mathbb{Z}/2\mathbb{Z})$ being a coarse moduli space for the space of circles in $\mathbb{R}^2$ centered at the origin.

(z) fine moduli space

[Beh14, Definition 1.17] Let $\mathscr{F}$ be a family of 'objects' in which there exists a notion of isomorphisms between objects. Write $\mathscr{F}/T$ for the family of objects that is continuously parametrized by the topological space T.

A **fine moduli space** is a space M such that

(i) the points of M are in one-to-one correspondence with isomorphism classes of the objects we're studying.

(ii) for every family $\mathscr{F}/T$ the associated moduli map $T \to M$ (which maps the point $t \in M$ to the isomorphism class of some member $\mathscr{F}_t$) is continuous.

(iii) every continuous map from a space T to M is a moduli map of some parametrized family by T.

(iv) if two families have the same moduli map, they are isomorphic families.

Example: In class we learned that the fine moduli space of circles in $\mathbb{R}^2$ centered at $(0,0)$ is $\mathbb{R}_{\geq 0}$. I try to match this with the above definition. In this case $\mathscr{F}$ is the family of circles and $M = \mathbb{R}_{\geq 0}$

(i) $\mathbb{R}_{\geq 0}$ is in one to one correspondence with the family of circles by identifying $r \in \mathbb{R}_{\geq 0}$ with the circle of radius r.

(ii) We need to properly define the moduli map to make this precise but intuitively we can imagine that if two circles are "close" then there radii will also be 'close'.

(iii) If $f : T \to \mathbb{R}_{\geq 0}$ is a continuous map then f continuously parametrizes a family of circles by mapping $t \in T$ to the circle of radius $f(t)$.

(iv) The same moduli map should define the same set of circles.

Very nice analysis of the arguments of the defn.

Part 2

In his paper *Moduli of sheaves: a modern primer*, [Lie17], Max Lieblich provides a new setting for the moduli theory of sheaves; a setting which treats sheaves and twisted sheaves as equal players. The motivation comes from recent developments in the theory of moduli spaces of sheaves, which rely heavily on the domain of so called 'twisted sheaves' - enough to suggest a more uniform view on sheaf theory.

This paper deals with the moduli spaces of sheaves and twisted sheaves, a sub-field of algebraic geometry that has many applications to other fields of mathematics. This is demonstrated by section 4 of Lieblich's paper, where he outlines a 'catalog of results'; a list of interesting theorems either about twisted sheaves or using twisted sheaves in fields such as category theory, non-commutative algebra and arithmetic.

The main goal of the paper, in Lieblich's words, is 'a more coherent theory that incorporates both sheaves and twisted sheaves as equals'. Lieblich hopes to achieve this with the theory of *merbes.* From what I can gather, a merbe is a generalisation of the concept of a gerbe that allows a more symmetric description of moduli problems. In his paper, Lieblich defines a merbe and a collection of associated concepts such as a *sheaf of merbes* and a *smooth proper curve-merbe*, essentially providing a way of describing moduli problems in a way that has equal emphasis on sheaves and twisted sheaves. In a sense, the main result of the paper is that *merbes are useful.* Lieblich demonstrates this with a series of examples and case studies which show that certain well-known results can be re-proven in the more general language of merbes. The two major case studies are the periodic-index problem for the Brauer group and the Tate conjecture for K3 surfaces.

In his introduction, Lieblich makes a long list of results and areas in mathematics which have made use of the moduli of sheaves. In the next paragraph he does a similar thing for twisted sheaves. This shows that individually the two subjects are extremely applicable, and hence already of high interest to the mathematical community. I would imagine that a theory with more coherence between sheaves and twisted sheaves, as described in this paper, could provide a more streamlined setting to talk about these problems and potentially highlight similarities and relationships between the theory of sheaves and twisted sheaves.

In Remark 6.5.16, Lieblich identifies a common proof idea that occurs a lot in the theory of moduli. The vague idea of the technique is that choosing a good compactification of the moduli problem leads to a certain hierarchical structure and playing different levels of the structure against each other leads to 'limiting theorems'. Such a hierarchical structure is also mentioned on page 10, where Lieblich talks about an implicit hierarchy in the 'second Chern class' and how it is possible to form a hierarchy of moduli spaces in which the boundary of a given space is made of spaces lower in the hierarchy. From this I can sort of gather that this hierarchical structure is an important aspect of the moduli theory of sheaves and may be a major reason for its applicability.

A highlight of Lieblich's paper is the large number of examples, allowing the reader to get a feel for the definitions and theorems first hand. The examples seem to be quite varied as well, which I think in addition to helping the reader's understanding, would help convince them that the material has broad applications. My only complaint about the formatting of the paper is that the titles for the different sections were quite small, meaning that if you wanted to quickly find, for example 'Part 2: A thought experiment', it may be hard to find when quickly skimming through the paper. Another thing is that a lot of the proofs in the paper were actually 'proof ideas', which may be annoying if I wanted to know the full proof of a result. However, I think that this is in line with the style of the paper; it is intended to give an overview of the useful applications of moduli spaces of (twisted) sheaves and merbes, so the author likely didn't want to draw too much attention to the details of proofs. Sometimes he did provide a reference to a full proof, which is useful but could've been done more.

This paper was first submitted as a contribution to the proceedings of the 2015 AMS Summer Institute in Algebraic Geometry. At the time Max Lieblich would've been either a professor or associate professor at the University of Washington (he was promoted in 2015). This means that at the time of writing this paper Lieblich had over a decade of experience as a mathematician. In addition to algebraic geometry, Lieblich is interested in education technology, computer vision and data science. His webpage is `https://max.lieblich.us/`.

Overall I get the sense that this paper is less about any particular result and is more about introducing a new way of talking about things that have already been talked about. The advantage is that this new language better expresses the direction that the field is heading in and better equips mathematicians to move forward.

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