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MAST90097 Assignment 1

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This document serves two purposes, and comes in two parts. The first purpose is to achieve marks in the mathematics subject Algebraic Geometry. The first part consists of a glossary of algebraic geometry terms, paired with a list of illustrative examples. The second part gives a summary of a modern paper in algebraic geometry, to show where the cutting edge of mathematics is, and how close we are getting to it through our coursework. The second purpose of this document is for my own personal benefit and education. I plan on keeping this file long after it has been submitted and assessed, as something I can look at in six or so years' time as a primer if I ever need to quickly relearn algebraic geometry. To this end, I have been informal with my speech at times, explaining things in the way that I understand them best rather in the way that is the most rigorous. I believe that this second goal - to actually understand the subject matter - is more important than the first, so I am willing to sabotage some marks in order to achieve this.[1]

Very honest, and well reasoned, this deserves extra marks.

1 Definitions

a. A **topological space** is a set X together with $\mathcal{T}_X$, a set of subsets of X that satisfies the following properties:

   i. $\emptyset \in \mathcal{T}_X$ and $X \in \mathcal{T}_X$.

   ii. If $U_1 \in \mathcal{T}_X$ and $U_2 \in \mathcal{T}_X$ then $U_1 \cap U_2 \in \mathcal{T}_X$.[2]

   iii. If $S \subseteq \mathcal{T}_X$ then $\bigcup_{V \in S} V \in \mathcal{T}_X$.

   The set $\mathcal{T}_X$ is usually called the **topology on X**

   Elements of $\mathcal{T}_X$ are called **open sets**.

   Example: A classic favourite example is the two-point set $X = \{0, 1\}$, together with the topology $\mathcal{T}_X = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$, the powerset of the two-point set (the set of all subsets of X). $\mathcal{T}_X$ clearly contains the empty set and X itself, so (i) is satisfied. Any intersection of subsets of X will still be a subset of X, so (ii) is satisfied. Finally, any union of subsets of X will still be a subset of X, so (iii) is satisfied.

   More generally, any set together with its powerset is a topological space. These such spaces are called ***discrete topologies***.

   Best example for this course is $\mathbb{C}$, probably. Need an example where are there a generating set for the topology.

b. A **metric space** is a set X together with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ that satisfies the following properties:

   i. If $x \in X$ then $d(x, x) = 0$.

   ii. If $x, y \in X$ and $x \neq y$ then $d(x, y) > 0$.

   iii. If $x, y \in X$ then $d(x, y) = d(y, x)$.

   iv. (Triangle inequality) If x, y, z are all in X then $d(x, z) \leq d(x, y) + d(y, z)$.

   Example: The vector space $\mathbb{R}^2$ forms a metric space with the following distance function:

$$d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$$
$$((x_1, y_1), (x_2, y_2)) \mapsto \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$$

$= \mathbb{C}$

State for $\mathbb{C}$ for this course

[1] P.S. Please do not confuse my footnotes for indices!

[2] In lectures a different definition is given - that finite intersections of open sets give open sets. This statement is equivalent to the one I give here. To prove one direction of this equivalence, take the case where $n = 2$, and to prove the other direction use induction. I prefer this definition because it feels cleaner somehow.

Thinking through, stating the reasons, changing the standard to a better one, for this deserves extra marks

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To verify (i), see that $d((x_1,y_1),(x_1,y_1)) = \sqrt{(x_1-x_1)^2+(y_1-y_1)^2} = \sqrt{0+0} = 0$.

To verify (ii), see that if $(x_1,y_1) \neq (x_2,y_2)$ *then both* $(x_1-x_2)^2$ *and* $(y_1-y_2)^2$ *are greater or equal to zero, with at least one of these values ~~being strictly greater than zero~~. This implies that their sum is strictly greater than zero, which implies that the square root of their sum is strictly greater than zero.*

do you know how to prove this?

To verify (iii), simply note that for any two real numbers a and b, $(a-b)^2 = (b-a)^2$.

Showing that the triangle inequality (iv) holds for this distance function is harder. To prove it we will prove that $d((x_1,y_1),(x_3,y_3))^2 \leq \left(d((x_1,y_1),(x_2,y_2)) + d((x_2,y_2),(x_3,y_3))\right)^2$. *We will need the following lemma: that* $x_1x_2 + y_1y_2 \leq \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$.

Proof of lemma: *We will prove this by proving that* $(x_1x_2+y_1y_2)^2 \leq (x_1^2+y_1^2)(x_2^2+y_2^2)$.

$$\begin{aligned}
\text{RHS - LHS} &= (x_1^2+y_1^2)(x_2^2+y_2^2) - (x_1x_2+y_1y_2)^2 \\
&= (x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2) - (x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2) \\
&= x_1^2y_2^2 + y_1^2x_2^2 - 2x_1y_2y_1x_2 \\
&= (x_1y_2 - y_1x_2)^2 \geq 0
\end{aligned}$$

So RHS $\geq$ *LHS.*

Main proof:

$$\begin{aligned}
\text{LHS} &= d((x_1,y_1),(x_3,y_3))^2 \\
&= (x_1-x_3)^2 + (y_1-y_3)^2 \\
&= (x_1-x_2+x_2-x_3)^2 + (y_1-y_2+y_2-y_3)^2 \\
&= (x_1-x_2)^2 + 2(x_1-x_2)(x_2-x_3) + (x_2-x_3)^2 + (y_1-y_2)^2 + 2(y_1-y_2)(y_2-y_3) + (y_2-y_3)^2 \\
&\leq (x_1-x_2)^2 + (y_1-y_2)^2 + 2\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\sqrt{(x_2-x_3)^2+(y_2-y_3)^2} + (x_2-x_3)^2 + (y_2-y_3)^2 \\
&\quad \text{(from lemma)} \\
&= \left(\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} + \sqrt{(x_2-x_3)^2+(y_2-y_3)^2}\right)^2 \\
&= \left(d((x_1,y_1),(x_2,y_2)) + d((x_2,y_2),(x_3,y_3))\right)^2 = \text{RHS}
\end{aligned}$$

c. A **ringed space** $(X,\mathcal{T}_X,\mathcal{O}_X)$ is a topological space $(X,\mathcal{T}_X)$ together with a sheaf of rings $\mathcal{O}_X$ over our topological space.

A **sheaf of rings** over $(X,\mathcal{T}_X)$ is a category of rings $\mathcal{O}_X$ together with a functor $\mathcal{F}: \mathcal{T}_X \to \mathcal{O}_X$ that satisfies the following properties:

defn of a functor

i. If $U \in \mathcal{T}_X$, $V \in \mathcal{T}_X$ and $U \subseteq V$, there exists a morphism $\text{res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$.

ii. If $U \in \mathcal{T}_X$, $\text{res}_U^U = \text{id}_{\mathcal{F}(U)}$.

iii. If $U \in \mathcal{T}_X$, $V \in \mathcal{T}_X$ and $W \in \mathcal{T}_X$, and $U \subseteq V \subseteq W$, then $\text{res}_U^V \circ \text{res}_V^W = \text{res}_U^W$.

defn of exact functor

iv. (Locality) If $U \in \mathcal{T}_X$, and $\mathcal{S} \subset \mathcal{T}_X$ is an **open cover** of U, and if $f_1, f_2 \in \mathcal{F}(U)$ such that $\text{res}_V^U(f_1) = \text{res}_V^U(f_2)$ for all $V \in \mathcal{S}$, then $f_1 = f_2$.

v. (Gluing) If $U \in \mathcal{T}_X$, and $\mathcal{S} \subset \mathcal{T}_X$ is an open cover of U, and for each $V \in \mathcal{S}$ there exists a $f_v \in \mathcal{F}(V)$ such that $\text{res}_{V\cap W}^V(f_v) = \text{res}_{V\cap W}^W(f_w)$ for any pair of sets $V, W \in \mathcal{S}$, then there exists an $f \in \mathcal{F}(U)$ such that $\text{res}_V^U(f) = f_v$ for all $V \in \mathcal{S}$. [3]

good

Remark: A category of rings $\mathcal{O}_X$ together with a functor that only satisfies the first three of these properties is called a ~~presheaf~~. Presheaves are useful objects in category theory.

Example: any topological ~~space can be turned into a ringed space~~. For concreteness, let's consider my favourite topological space: three-dimensional the torus, $S^1 \times S^1 \times S^1$. *Assign to it a sheaf of rings by associating to each*

[3] Again, this differs from the definition supplied in class. I present this definition here because the other one suffers from the disadvantage that I cannot understand it.

why 3-dimensional?

from the in class defn it looks like the part you don't understand is ker = im?

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This word makes me feel stupid. Is it really necessary here?

open set U the ring of continuous functions from U to $\mathbb{R}$ $\boxed{\text{denoted by } C^0(U)}$. If $U \subseteq V$ then res^V_U maps $C^0(V)$ to $C^0(U)$ by mapping any continuous function from V to $\mathbb{R}$ to its restriction onto U, so the first property is satisfied. Clearly, any function $f : U \to \mathbb{R}$ restricted to U is still the same function so (ii) is satisfied, and if $U \subseteq V \subseteq W$ and $g : W \to \mathbb{R}$ then restricting g to U is the same as restricting g to V and then restricting $g|_V$ to U, so (iii) is satisfied. So this collection of rings is at least a presheaf.

To show that locality is satisfied, let f_1 and f_2 be two functions in $C^0(U)$, and let S be an open cover of U. If the restriction of each of these two functions are the same on each set $V \in S$, then the value of these functions must be exactly the same on each point in $\bigcup_{V \in S} V$, which contains U. So these functions must have the same value on each point in U, and hence must be equal.

To show that gluing is satisfied, let $U \in \mathcal{T}_X$, and $S \subseteq \mathcal{T}_X$ be an open cover of U, and choose functions $f_v \in C^0(V)$ for each $V \in S$ such that any such pair of functions agree on the intersection of their domains — that is, if $V, W \in S$ then $f_v|_{V \cap W} = f_w|_{V \cap W}$. We can then define a function $f : U \to \mathbb{R}$ that agrees pointwise with all of these functions — if $x \in U$ there exists a $V \in S$ such that $x \in V$, so define $f(x) = f_v(x)$. This function is well defined, because if $x \in V$ and $x \in W$, where $V, W \in S$ then $f_w(x) = f_v(x)$, because $x \in V \cap W$ and these two functions agree on the intersection of their domains.

To show that f is continuous, let $W \subseteq \mathbb{R}$ be an open set. We need to show that the preimage of this set $f^{-1}(W)$ is open. Because S is an open cover of U, $f^{-1}(W) = \bigcup_{V \in S} f|_V^{-1}(W)$. Now, since each $f|_V = f_v$ is continuous, $f|_V^{-1}(W)$ is open for each $V \in S$. So $f^{-1}(W) = \bigcup_{V \in S} f|_V^{-1}(W)$ is open. So f is a continuous function. This shows that by gluing together a collection of continuous functions that agree on the intersection of their domains, and if the domains of these functions together cover U, we get a well-defined continuous function that is defined for all of U. This also shows that our given functor defines a structure sheaf for a topological space.

d. Let K be a fixed algebraically closed field. The **affine n-space over** K is the set of all n-tuples of elements of K.[4] Often this space comes equipped with the Zariski topology and the sheaf of regular functions, making it a ringed space.

Let K be a fixed algebraically closed field. Let A^n be an affine n-space over K. The **Zariski topology on** A^n, which we denote by $\mathcal{T}^{Zar}$, is the set of all subsets of A^n whose complements are algebraic. A set $Y \subseteq A^n$ is **algebraic** if there exists a family of polynomials $T \subseteq K[x_1, \ldots, x_n]$ such that for all $f \in T$, $f(p) = 0$ iff $p \in Y$.

Say we have an affine n-space A^n over an algebraically closed field K, and set U that is open in A^n when it is given the Zariski topology. A function $\varphi : U \to K$ is **regular on** U if, for every $a \in U$ there exists a set $U_a \in \mathcal{T}^{Zar}$ such that $a \in U_a$ and $U_a \subseteq U$, and a pair of polynomials $f, g \in K[x_1, \ldots, x_n]$ such that if $x \in U_a$, $f(x) \neq 0$, and $\varphi(x) = \frac{g(x)}{f(x)}$.

For any open set $U \in \mathcal{T}^{Zar}$ the set of all regular functions on U forms a ring. The sheaf of regular functions maps each open U set to the set of functions that are regular on U, and this forms a sheaf on our affine space, making it into a ringed space. Restriction maps work exactly how you'd expect: if $V \subseteq U$ then $\mathrm{res}^U_V(f) = f|_V$.

Example: My favourite algebraically closed field is $\mathbb{C}$ and my favourite number is 9. So let's give $\mathbb{C}^9$ as an example of an affine space

Here I would like to pause and, for my own benefit, step outside the austere struture of the assignment to supply two proofs: that an affine space with the Zariski topology is indeed a topological space, and that the sheaf of regular functions really is a sheaf of rings. I'll even do it in proof machine.

Let K be a fixed algebraically closed field. Let A^n be the affine n-space over K.

To show:

(i) $\mathcal{T}^{Zar}$ is a topology on A^n.

(ii) For every $U \in \mathcal{T}^{Zar}$, the set of regular functions on U, denoted by $\mathcal{O}(U)$, is a ring.

(iii) The sheaf of regular functions $\mathcal{O}$ is a sheaf of rings over $(A^n, \mathcal{T}^{Zar})$.

i. To show: $\mathcal{T}^{Zar}$ is a topology on A^n.

[4] This definition comes from Hartshorne. The one on Wikipedia is different. and the one on Harder?

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i. To show: $\emptyset \in \mathcal{T}^{Zar}$ and $A^n \in \mathcal{T}^{Zar}$.
Consider the polynomials $0 \in K[x_1, \dots, x_n]$ and $1 \in K[x_1, \dots, x_n]$. The vanishing set of 0 is the entire set A^n, so its complement is the empty set. So the empty set is in the Zariski topology. The vanishing set of 1 is the empty set, so its complement is the entire set A^n. So the entire set A^n is in the Zariski topology.

ii. To show: if $U_1 \in \mathcal{T}^{Zar}$ and $U_2 \in \mathcal{T}^{Zar}$ then $U_1 \cap U_2 \in \mathcal{T}^{Zar}$.
Assume $U_1 \in \mathcal{T}^{Zar}$ and $U_2 \in \mathcal{T}^{Zar}$. Then we know there exists two sets of polynomials $S_1, S_2 \subseteq K[x_1, \dots, x_n]$ such that $U_1 = \{(a_1, \dots, a_n) \in A^n \mid \text{if } f \in S_1 \text{ then } f(a_1, \dots, a_n) \neq 0\}$, and $U_2 = \{(a_1, \dots, a_n) \in A^n \mid \text{if } f \in S_2 \text{ then } f(a_1, \dots, a_n) \neq 0\}$. Then $U_1 \cap U_2 = \{(a_1, \dots, a_n) \in A^n \mid \text{if } f \in S_1 \cup S_2 \text{ then } f(a_1, \dots, a_n) \neq 0\}$. So $U_1 \cap U_2$ is the vanishing set of the set of polynomials $S_1 \cup S_2 \subseteq K[x_1, \dots, x_n]$. So $U_1 \cap U_2 \in \mathcal{T}^{Zar}$.

iii. To show: if $\mathcal{S} \subseteq \mathcal{T}^{Zar}$ then $\bigcup_{V \in \mathcal{S}} V \in \mathcal{T}^{Zar}$.
For each $V \in \mathcal{S}$ there is a set of polynomials $Q_V \subseteq K[x_1, \dots, x_n]$ such that $V = \{(a_1, \dots, a_n) \in A^n \mid \text{if } f \in Q_V \text{ then } f(a_1, \dots, a_n) \neq 0\}$. So $\bigcup_{V \in \mathcal{S}} V = \{(a_1, \dots, a_n) \in A^n \mid \text{if } f \in \bigcap_{V \in \mathcal{S}} Q_V \text{ then } f(a_1, \dots, a_n) \neq 0\}$. So $\bigcup_{V \in \mathcal{S}} V$ is the vanishing set of the set of polynomials $S_1 \cup S_2 \subseteq K[x_1, \dots, x_n]$.
So $\bigcup_{V \in \mathcal{S}} V \in \mathcal{T}^{Zar}$.

So the Zariski topology is a topology for A^n.

ii. To show: for every $U \in \mathcal{T}^{Zar}$, the set of regular functions on U, denoted by $\mathcal{O}(U)$, is a ring.
Define addition and multiplication as follows: Let $\psi_1, \psi_2 : U \to K$ be regular on U. Then:

$$\begin{aligned} \psi_1 + \psi_2 : U &\to K \\ x &\mapsto \psi_1(x) + \psi_2(x) \end{aligned}$$

$$\begin{aligned} \psi_1 \cdot \psi_2 : U &\to K \\ x &\mapsto \psi_1(x) \cdot \psi_2(x) \end{aligned}$$

We want to show that these two new functions are still regular.
Let $a \in U$. We know that there exists two open sets $U_{a1}, U_{a2} \subseteq U$ such that $a \in U_{a1}, a \in U_{a2}$ and four polynomials $f_1, f_2, g_1, g_2 \in K[x_1, \dots, x_n]$ such that if $x \in U_{ai}, f_i(x) \neq 0$ and $\psi_i(x) = \frac{g_i(x)}{f_i(x)}$.
Let $U_a = U_{a1} \cap U_{a2}$. This will be an open set because the Zariski topology is a topology, it will be a subset of U because both U_{a1} and U_{a2} are subsets of U, and it will contain a because both U_{a1} and U_{a2} contain a. Now, if $x \in U_a, f_1(x) \neq 0, f_2(x) \neq 0$ and $(\psi_1 + \psi_2)(x) = \frac{g_1(x)}{f_1(x)} + \frac{g_2(x)}{f_2(x)} = \frac{g_1(x)f_2(x) + g_2(x)f_1(x)}{f_1(x)f_2(x)}$. So locally $\psi_1 + \psi_2$ is a quotient of polynomials. So $\psi_1 + \psi_2$ is regular. Similarly, if $x \in U_a, f_1(x) \neq 0, f_2(x) \neq 0$ and $(\psi_1 \cdot \psi_2)(x) = \frac{g_1(x)}{f_1(x)} \cdot \frac{g_2(x)}{f_2(x)} = \frac{(g_1 \cdot g_2)(x)}{(f_1 \cdot f_2)(x)}$. So locally $\psi_1 \cdot \psi_2$ is a quotient of polynomials. So $\psi_1 \cdot \psi_2$ is regular.
The multiplicative identity in this ring is the function that sends everything in U to 1. The additive identity in this ring is the function that sends everything in U to 0. The additive inverse of any function in $\mathcal{O}(U)$ can be constructed by multiplying the function in question by the function that sends everything in U to -1. All other ring axioms are inhereted by the properties of our field, K.

iii. To show: the sheaf of regular functions $\mathcal{O}$ is a sheaf of rings over $(A^n, \mathcal{T}^{Zar})$.
To see it is a presheaf is easy from the definition of restriction maps. Localilty is also easy because if two functions are equal on each point, they are equal. The tricky part is gluing. We wish to show that if $U \in \mathcal{T}^{Zar}$, and $\mathcal{S} \subset \mathcal{T}^{Zar}$ is an open cover of U, and for each $V \in \mathcal{S}$ there exists a $f_v \in \mathcal{O}(V)$ such that $\mathrm{res}^V_{V \cap W}(f_v) = \mathrm{res}^W_{V \cap W}(f_w)$ for any pair of sets $V, W \in \mathcal{S}$, then there exists an $f \in \mathcal{O}(U)$ such that $\mathrm{res}^U_V(f) = f_v$ for all $V \in \mathcal{S}$.
Assume $U \in \mathcal{T}^{Zar}$.
Assume $\mathcal{S} \subset \mathcal{T}^{Zar}$ is an open cover of U, and for each open set $V \in \mathcal{S}$ choose regular functions $f_v \in \mathcal{O}(V)$ such that any pair of functions agree on the intersection of their domains - that is, $\mathrm{res}^V_{V \cap W}(f_v) = \mathrm{res}^W_{V \cap W}(f_w)$ for any pair of sets $V, W \in \mathcal{S}$.
We can then define a function $f : U \to K$ that agrees pointwise with all of these functions - if $x \in U$ there exists a $V \in \mathcal{S}$ such that $x \in V$, so let $f(x) = f_v(x)$. This will be well-defined because if $x \in V$ and $x \in W$ where $V, W \in \mathcal{S}$ then $x \in V \cap W$, so $f_v(x) = f_w(x)$. So choosing either $f(x) = f_v(x)$ or $f(x) = f_w(x)$ gives us the same result.

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Our last step is to show that this function is still regular - that is, if $a \in U$ then there exists an open set $U_a \subseteq U$ such that $a \in U_a$ and a pair of polynomials $p, q \in K[x_1, \ldots, x_n]$ such that if $x \in U_a$ then $q(x) \neq 0$ and $f(x) = \frac{p(x)}{q(x)}$.

Assume $a \in U$. Then we know there exists a $V \in S$ such that $a \in V$, because S covers U. We also know that $\mathrm{res}^U_V(f) = f_v$ and f_v is regular. Because f_v is regular, we know that there exists an open set $U_a \subseteq V$ such that $a \in V_a$ and a pair of polynomials $p, q \in K[x_1, \ldots, x_n]$ such that if $x \in U_a$ then $q(x) \neq 0$ and $f_v(x) = \frac{p(x)}{q(x)}$. But $x \in V$, so $f(x) = f_v(x)$, so $f(x) = \frac{p(x)}{q(x)}$, too. So f is regular.

This proves that an affine space, with the Zariski topology and the set of regular functions, is a ringed space.

Yes, thank you. I very much appreciate this good mathematics proof.

e. Let K be a fixed algebraically closed field. A **projective n-space over** K is the set of all n-tuples of elements of K (except for $(0, \ldots, 0)$) under the equivalence relation given by $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in K \setminus \{0\}$. This can also be made into a ringed space by taking the topology to be the Zariski topology and sheaf of regular functions as before, but restricting our polynomials to be homogenous so that $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$ for some $d \in \mathbb{Z}_{\geq 0}$. (A **homogenous** polynomial is one where every term has the same degree - for example $x^2 + 2xy - 3zx$.)

Why? do we do this?

Example: Stay in $\mathbb{C}$, but large numbers are now pretty unwieldly, so let's look at $\mathbb{P}^1$, the projective space defined by $\mathbb{C}^2 \setminus \{(0,0)\}$ under the equivalence relation that identifies $(a_0, a_1) \sim (\lambda a_0, \lambda a_1)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Every element of P^2 is equivalent to exactly one element in the set $\{(1, z) \mid z \in \mathbb{C}\} \cup \{(0,1)\}$. To see this, let $(a_0, a_1) \in \mathbb{C}^2$. If $a_0 \neq 0$, this pair is equivalent to $(1, a_0^{-1} a_1)$. Otherwise, if $a_0 = 0$ and $a_1 \neq 0$, this pair is equivalent to $(0, 1)$ by multiplying both numbers by a_1^{-1}.

But why/how is it a ringed space?

f. An **affine variety** is an irreducible closed subspace of an affine space with the Zariski topology.

Let $(X, \mathcal{T}_X)$ be a topological space. Let $Y \subseteq X$. Y is **irreducible** if it is nonempty, closed, and cannot be expressed as the union of any two proper subsets, each of which is closed in Y (when Y is given the subspace topology).

Why do you want irreducible?

Example: Let $K = \mathbb{C}$ be our algebraically closed field. Let $f = x^2 + y^2 - 1 \in \mathbb{C}[x, y]$, and let $T = \{f\}$. Define $Z(f) = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$. Now, $\mathbb{C}^2$ is an affine space and $Z(f)$ is algebraic, so $Z(f)$ is a closed subset of $\mathbb{C}^2$ when it is given the Zariski topology. We claim that $Z(f)$ is irreducible - that is, there do not exist any two algebraic proper subsets of $Z(f)$ such that $Z(f) = Y_1 \cup Y_2$. I don't know how to prove this is true precisely, but I know the idea is that (i) f is irreducible in $K[x, y]$ and (ii) finding two algebraic proper subsets of $Z(f)$ such that $Z(f) = Y_1 \cup Y_2$ is equivalent to finding two polynomials in $K[x, y]$ that multiply together to give f.

restrict Spec to quotients of polynomial rings.

If you want an example I can actually prove, take $g(x) = x - 2$ and $Z(g) = \{x \in \mathbb{C} \mid g(x) = 0\}$. This set consists of a single point and is closed in $\mathbb{C}$ when it is given the Zariski topology, and it clearly cannot be described as the union of two proper subsets, because the only proper subset of a set with only one element is the empty set. So $Z(g) = \{2\}$ is an affine variety in $\mathbb{C}$. But it is not a very interesting one.

Is 2 disjoint pts (roots of $(x-2)(x-1)$) not an affine variety?) Why not?

g. A **variety** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine variety.

Two ringed spaces $(X, \mathcal{T}_X, \mathcal{O}_X), (Y, \mathcal{T}_Y, \mathcal{O}_Y)$ are **isomorphic** if there exists a **homeomorphism** $\Phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ such that for every $U \in \mathcal{T}_X$, there exists a **ring isomorphism** $\varphi_U : \mathcal{O}_X(U) \to \mathcal{O}_Y(\Phi(U))$.

A ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ is **locally isomorphic** to a second ringed space $(Y, \mathcal{T}_Y, \mathcal{O}_Y)$ if for each $p \in X$ there exists a $U \in \mathcal{T}_X$ such that $x \in U$, together with an open set $V \in \mathcal{T}_Y$ and an isomorphism $\Phi : (U, \mathcal{T}_U, \mathcal{O}_U) \to (V, \mathcal{T}_V, \mathcal{O}_V)$.

In other words, a **variety** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that satisfies the following properties:

i. X has a finite open covering $\{U_\alpha\}$ such that each $(U_\alpha, \mathcal{T}_\alpha, \mathcal{O}_X(U_\alpha))$ is isomorphic to an affine algebraic variety, where $\mathcal{T}_\alpha$ represents the subspace topology of X on U_α.

ii. $(X, \mathcal{T}_X, \mathcal{O}_X)$ satistfies the **separation axiom**: $\{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$ when this set is equipped with the Zariski topology.

Example: $(\mathbb{C}^2, \mathcal{T}^{Zar}, \mathcal{O})$ is an affine variety. To prove this, we have to prove that $\mathbb{C}^2$ is irreducible in the Zariski topology - that is, there does not exist two closed proper sets $Y_1, Y_2 \subset \mathbb{C}^2$ such that $\mathbb{C}^2 = Y_1 \cup Y_2$. This is equivalent to proving that there does not exist any two (nonconstant) polynomials in $f, g \in \mathbb{C}[x, y]$ whose

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vanishing sets cover $\mathbb{C}$ *- ie. if* $(x,y) \in \mathbb{C}^2$ *then* $f(x,y) = 0$ *or* $g(x,y) = 0$. *The qualifier that our polynomials are nonconstant is equivalent to saying that our closed sets are neither* $\mathbb{C}^2$ *nor* $\emptyset$. *We can re-write this as saying our polynomials are of degree at least 1.*

This proof will precede using a contradiction argument. Assume that there exists two polynomials of degree at least 1 $f, g \in \mathbb{C}[x,y]$ *such that if* $(x,y) \in \mathbb{C}^2$ *then* $f(x,y) = 0$ *or* $g(x,y) = 0$. *This implies that if* $y = 1$ *then* $f(x,1) = 0$ *or* $g(x,1) = 0$ *for all* $x \in \mathbb{C}$. *However, because* f *and* g *are polynomials, the equations* $f(x,1) = 0$ *and* $g(x,1) = 0$ *must have only finite solutions each - in fact,* $f(x,1) = 0$ *has at most* $deg(f)$ *solutions. This would imply that there are only finitely many complex numbers, which gives us a contradiction. So* $\mathbb{C}^2$ *is irreducible in the Zariski topology, and consequently* $(\mathbb{C}^2, \mathcal{T}^{Zar}, \mathcal{O})$ *is an affine variety. All affine varieties are isomorphic (and hence locally isomorphic) to themselves, so this space is also a variety.*

$\mathbb{P}^1$, *the projective space defined in part (e), is also a variety (that is not affine). To do this, we cover it by the open sets* $\mathbb{P}^1 \setminus \{[0:1]\}$ *and* $\mathbb{P}^1 \setminus \{[1:0]\}$, *and prove that each of them are isomorphic to* $(\mathbb{C}, \mathcal{T}^{Zar}, \mathcal{O})$. *(This is an affine variety for similar reasons to the proof I gave above). We will give explicitly an isomorphism between* $\mathbb{P}^1 \setminus \{[0:1]\}$ *and* $\mathbb{C}$; *the other one can be constructed by concatenating this transform with the reflection isomorphism* $[x:y] \mapsto [y:x]$.

$$\Phi : \mathbb{P}^1 \setminus \{[1:0]\} \to \mathbb{C}$$
$$[a_0 : a_1] \mapsto a_0 a_1^{-1}$$

$$\Phi^{-1} : \mathbb{C} \to \mathbb{P}^1 \setminus \{[1:0]\}$$
$$z \mapsto [z:1]$$

To show this is a homeomorphism, we will show that closed sets map to closed sets under the image of this transformation. The Zariski topology of $\mathbb{C}$ *is generated by the complements of the zero sets of polynomials of degree one - polynomials of the form* $f(z) = z - c$. *Under the transformation of* Φ^{-1} *this polynomial looks like* $a_0 - ca_1$. *So polynomials in* $\mathbb{C}$ *map to homogenous polynomials in* $\mathbb{P}^1$. *So the zero sets of polynomials in* $\mathbb{C}$ *map the zero sets of homogenous polynomials in* $\mathbb{P}^1$. *So closed sets map to closed sets. So this is continuous. You can prove that the inverse is also continuous with a similar argument.*

Let $U \in \mathbb{P}^1 \setminus \{[1:0]\}$ *be open. The pullback of* $\Phi^{-1}|_U$ *gives a ring homomorphism from* $\mathcal{O}(U)$ *to* $\mathcal{O}(\Phi(U))$.

h. An **affine scheme** is an element of the image of Spec.

Spec is a functor that sends commutative rings to ringed spaces.

Let R be a commutative (unital) ring. $\mathrm{Spec}(R) := (X, \mathcal{T}_X^{Zar}, \mathcal{O}_X)$, where X is the set of all **prime ideals** in R. $\mathcal{T}_X^{Zar}$ is the Zariski topology on X, and $\mathcal{O}_X$ is the structure sheaf on X.

The **Zariski topology** on X is defined to be the set $\mathcal{T}_X^{Zar} := \{X \setminus V(S) \mid S \subseteq R\}$, where $V(S) := \{p \in X \mid \text{if } x \in S \text{ then } x \in p\}$.

The structure sheaf for this ringed space is defined on the basic sets X_g in the Zariski topology. For any $g \in R$, $X_g := \{p \in X \mid g \notin p\}$, and our sheaf maps X_g to the basic ring $R[\frac{1}{g}] := \{\frac{f}{g^k} \mid f \in R, k \in \mathbb{Z}_{\geq 0}\}$. Operations on our basic ring are defined as follows:

$$+ : R[\frac{1}{g}] \times R[\frac{1}{g}] \to R[\frac{1}{g}]$$
$$(\frac{f_1}{g^k}, \frac{f_2}{g^l}) \mapsto \frac{f_1 g^l + f_2 g^k}{g^{k+l}}$$

$$\cdot : R[\frac{1}{g}] \times R[\frac{1}{g}] \to R[\frac{1}{g}]$$
$$(\frac{f_1}{g^k}, \frac{f_2}{g^l}) \mapsto \frac{f_1 f_2}{g^{k+l}}$$

and the following equivalence relation is imposed: $\frac{f_1}{g^k} = \frac{f_2}{g^l}$ if there exists an $n \in \mathbb{Z}_{\geq 0}$ such that $f_1 g^{l+n} = f_2 g^{k+n}$.

Restriction maps in this scheme work as follows. The statement $X_k \subseteq X_g$ is equivalent to the statement "if $p \in X_k$ then $p \in X_g$", which is equivalent to "if $k \notin p$ then $g \notin p$", which is equivalent to "if $g \in p$ then $k \in p$". This condition is satisfied if and only if there exists an $s \in R$ such that $k = sg$.

Morally, when we restrict X_g to X_k, we want to replace the denominator g^n with the denominator $k^n = (sg)^n$. To keep this equal, however, we also must multiply the top by s^n. This gives us the definition of restriction maps:

$$\mathrm{res}^{X_g}_{X_k} : R[\frac{1}{g}] \to R[\frac{1}{k}]$$
$$\frac{f}{g^n} \mapsto \frac{fs^n}{k^n}$$

Example: Take our commutative ring to be $\mathbb{Z}$. The set of all prime ideals of $\mathbb{Z}$ are ideals of the form $p\mathbb{Z} = \{pa \mid a \in \mathbb{Z}\}$, where $p \in \mathbb{Z}$ is a prime number. For this example, the Zariski topology can be interpreted as the set of all sets with finite complements - given any finite set of prime ideals $p_1\mathbb{Z}, \ldots, p_n\mathbb{Z}$ we wish to exclude from our open set, simply multiply the corresponding prime numbers together, and you will find that $V(p_1 \cdots p_n\mathbb{Z})$ is a closed set in the Zariski topology.
In the integers, every ideal is a principal ideal, and for any subset $S \subseteq R$, $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by S. This means we can think of the Zariski topology as $\{R \setminus V(\langle a\rangle) \mid a \in \mathbb{Z}\}$. Using this interpretation, our sheaf maps each open set $R \setminus V(\langle a\rangle)$ to the ring $\mathbb{Z}[\frac{1}{a}]$, the subset of the rational numbers for which the denominator can be expresssed as a power of a.
As an example of a restriction map, let's look at $6\mathbb{Z} \subset 2\mathbb{Z}$:

$$\mathrm{res}^{2\mathbb{Z}}_{6\mathbb{Z}} : \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{6}]$$
$$\frac{a}{2^n} \mapsto \frac{a3^n}{6^n}$$

This shows us how any fraction with a power of 2 on the denominator can be made into a fraction with a power of 6 on the denominator, allowing us to "include" $\mathbb{Z}[\frac{1}{2}]$ into $\mathbb{Z}[\frac{1}{6}]$.

i. A **scheme** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme.

Example: our a priori mentioned projective space example $\mathbb{P}^1 = \mathbb{C}^2 \setminus \{(0,0)\}/(a_0, a_1) \sim (\lambda a_0, \lambda a_1)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ is also an example of a scheme! This is because both $\mathbb{P}^1 \setminus \{[(1,0)]\}$ and $\mathbb{P}^1 \setminus \{[(0,1)]\}$ are open sets in $\mathbb{P}^1$ that are isomorphic to $\mathbb{C}$, which is an affine scheme because $Spec(\mathbb{C}[x]) \cong (\mathbb{C}, \mathcal{T}^{Zar}, \mathcal{O})$.

j. A **topological manifold** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine topological manifold.

An **affine topological manifold** is a ringed space of the form $(\mathbb{R}^n, \mathcal{T}^{std}, C^0)$, where $\mathcal{T}^{std}$ is the standard topology on $\mathbb{R}^n$ and C^0 is the sheaf of continuous functions from open subsets of $\mathbb{R}^n$ to $\mathbb{R}$.

Example: We're gonna go with a regular crowdpleaser, the two-dimensional torus, T^2. It can be defined in many ways, but my favourite is as a quotient space: $\mathbb{R}^2/\sim$, where the equivalence relation is generated by identifying $(x, y) \sim (x + n, y + m)$ for any pair of integers $n, m \in \mathbb{Z}$. The open sets on this space are the images of the open sets of $\mathbb{R}^2$ with the standard topology $\mathcal{T}^{std}$ under the quotient map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\sim$. To make this topological space a ringed space, pair it with C^0, the sheaf of continuous functions from open subsets of T^2 to $\mathbb{R}$
To show that this space is locally isomorphic to $(\mathbb{R}^2, \mathcal{T}^{std}, C^0)$, let $[x] \in T^2$ be any point on our torus. Then looking at the open set $\mathcal{B}_{\frac{1}{3}}([x])$, and fixing an individual $x \in \pi^{-1}([x])$, we restrict our quotient map to $\pi : \mathcal{B}_{\frac{1}{3}}(x) \to \mathcal{B}_{\frac{1}{3}}([x])$ to get an isomorphism.

k. A **smooth manifold** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine smooth manifold.

An **affine smooth manifold** is a ringed space of the form $(\mathbb{R}^n, \mathcal{T}^{std}, C^\infty)$, where $\mathcal{T}^{std}$ is the standard topology on $\mathbb{R}^n$ and C^∞ is the sheaf of continuous, infintely differentiable functions from open subsets of $\mathbb{R}^n$ to $\mathbb{R}$.

The previous example, the torus, can also be made into a smooth manifold. If you were reading a textbook by John M. Lee, you would say this is done by giving the torus a smooth structure - i.e. constructing an atlas of smooth coordinate charts that agree up to diffeomorphism on their intersections. But we're not your regular differential geometers. We are algebraic geometers. So instead we will say what is really meant - to turn our

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means? ok infinitely differentiable.

torus into a smooth manifold, we pair it with C^0, the sheaf of smooth functions from open subsets of T^2 to $\mathbb{R}$, and this becomes locally isomorphic to $(\mathbb{R}^n, \mathcal{T}^{std}, C^\infty)$ in an analogous way to the way described above.

good

l. Let $r \in \mathbb{Z}_{>0}$. A C^r **manifold** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine C^r manifold.

An **affine** C^r **manifold** is a ringed space of the form $(\mathbb{R}^n, \mathcal{T}^{std}, C^r)$, where $\mathcal{T}^{std}$ is the standard topology on $\mathbb{R}^n$ and C^r is the sheaf of continuous, r-times differentiable functions from open subsets of $\mathbb{R}^n$ to $\mathbb{R}$.

Example: to emphasise the analogy between topological manifolds, smooth manifolds and C^r manifolds, we will again take the torus as our topological space, but this time we will pair it with C^r, the sheaf of continuous, r-times differentiable functions from open subsets of T^2 to $\mathbb{R}$. Under the same local isomorphisms described in (j), this space will be locally isomorphic to $(\mathbb{R}^n, \mathcal{T}^{std}, C^r)$.
While these examples have been given with the purpose of explaining how these three definitions are similar, it is important to realise that no two of the three examples given, $(T^2, \mathcal{T}^{std}, C^0)$, $(T^2, \mathcal{T}^{std}, C^r)$, $(T^2, \mathcal{T}^{std}, C^\infty)$ are the same. This is beause they have different sheafs of functions lying over them. A good analogy for this comes from metric spaces. Define $d_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d_k((x_1, \dots, x_n), (y_1, \dots, y_n)) = \left(\sum_{i=1}^n |x_i - y_i|^k\right)^{\frac{1}{k}}$. Now two metric spaces $(\mathbb{R}^n, d_k)$ and $(\mathbb{R}^n, d_l)$ may look similar, but if they have different distance functions they cannot be considered the same space. It is the same with these three ringed spaces. While they are all defined on the same surface with the same topology, the sheaves are different, making the ringed space as a whole a different algebraic object.

Wonderful.

m. A **complex manifold** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affine complex manifold.

An **affine complex manifold** is a ringed space of the form $(\mathbb{C}^n, \mathcal{T}^{std}, \mathcal{C}^{hol})$, where $\mathcal{T}^{std}$ is the standard topology on $\mathbb{C}^n$ and $\mathcal{C}^{hol}$ is the sheaf of holomorphic functions from open subsets of $\mathbb{C}^n$ to $\mathbb{R}$.

Example: Of course I'm going to talk about the Reimann sphere. This guy is very similar to $\mathbb{P}^1$, defined in part (c), but this time he comes equipped with the sheaf of locally holomorphic functions as his sheaf. The local homeomorphisms between the underlying topological spaces are the same as those described in part (g), and the ring homomorphisms are again the pullbacks of these local homeomorphisms.

Can you define this precisely (and show that it is a complex manifold)

n. See (e).

o. A **CW complex** is a special kind of topological space that is defined inductively. Here we follow the writings of Hatcher in his book Algebraic Topology.

Format better, too difficult to parse

Begin by defining X^0, the zero skeleton of our space, to be a set of discrete points. Now we define the X^n skeleton from the X^{n-1} skeleton in the following way. Let $\phi_\alpha : S^{n-1} \to X^{n-1}$ be a family of continuous maps. Then $X^n = X^{n-1} \sqcup D^n_\alpha / \sim$, the disjoint union of our X^{n-1} skeleton with a closed n-dimensional unit disk for each α, under the equivalence relation $x \sim \phi_\alpha(x)$ for each $x \in S^{n-1} \subset D^n_\alpha$.

If there exists an $n \in \mathbb{Z}_{\geq 0}$ such that $X^m = X^n$ for all $m > n$, we define our CW-complex X to be its n-skeleton, $X := X^n$. Otherwise we define our CW complex to be $X := \bigcup_n X^n$. The topology on this space is the quotient topology induced by all of the attaching maps ϕ_α.

Arun said in lectures that the definitions of disks and quotient topologies are not obvious. I don't know if they are obvious or not - but I've read way too much Hatcher, so I'm obviously biased. The n-dimensional unit disk $D^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq 1\}$.

Let $(X, \mathcal{T}_X)$ be a topological space, Y be a set and $\pi : X \to Y$ be a surjective map. The **topology induced** by π is the set of sets $\mathcal{T}_Y := \{V \subseteq Y \mid \pi^{-1}(V) \in \mathcal{T}_X\}$. In other words, this is the largest topology on Y you can have such that the map π is continuous. If we call out surjective map a quotient map, then this topology is called the quotient topology, but really these two words are synonyms.

Not all engineers will be able to accurately gauge

yes.

Example: There are other, more peculiar examples of CW complexes, but I'd like to pay homage to travelling salesman, plane schedules and four-coloured maps. That's right: every graph is a 1-dimensional CW complex, and every 1-dimensional CW complex is a graph. The X^0 skeleton of a graph is its set of vertices, and its X^1 skeleton is all of its edges, consisting of copies of the interval $D^1 = [-1, 1] \subset \mathbb{R}$ with each of its endpoints identified with points in its set of vertices. Note that two ends of the same edge can be attached to the same vertex, making loops.

Most important for this class is PU[illegible] influence your boss

p. The **simplex category** Δ is the category of nonempty finite ordinals (ordered sets of the form $[n] = \{0, 1, \dots, n\}$ wheer $n \in \mathbb{Z}_{\geq 0}$) with morphisms given by order preserving maps.

Which are? Perhaps this reader doesn't know.

means?

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A **simplicial set** is a family of sets $X_\bullet = \{X_i\}, i \in \mathbb{Z}_{\geq 0}$ and of maps $X(f) : X_n \to X_m$ for each nondecreasing map $f : [m] \to [n]$ such that $X(\mathrm{id}) = (\mathrm{id})$, and $X(g \circ f) = X(f) \circ X(g)$. In other words, a simplicial set is a presheaf from the simplex category to the category of sets.

Example: We will look at the singulr simplices of a topological space. First we define the n-simplex in $\mathbb{R}^n$ *as follows:* $\Delta^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_i x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i\}$. *Now let X be a topological space. For each* $n \in \mathbb{Z}_{\geq 0}$, *define* X_n *to be the space of all continuous functions from* Δ^n *to X. Given a continuous map* $f : \Delta^m \to \Delta^n$ *we define* $X(f) : X_n \to X_m$ *to be the map that maps any continuous function* $\sigma : \Delta^n \to X$ *to* $\sigma \circ f : \Delta^m \to X$. *Clearly* $X(id)$ *maps each map to itself, and by definition* $X(g \circ f)(\sigma) = \sigma \circ (g \circ f) = (\sigma \circ g) \circ f = X(f)((\sigma \circ g)) = (X(f) \circ X(g))(\sigma)$ *for all* σ. *So letting* $X_\bullet = \{X_i\}, i \in \mathbb{Z}_{\geq 0}$, *we get a simplicial set.*

q. The **spectrum of a ring** R is the set of all prime ideals over that ring. only a set?

A **prime ideal** is an ideal I such that if $ab \in I$ then either $a \in I$ or $b \in I$.

Example: Take our ring to be $\mathbb{Z}$, *the ring of all integers. The set of all prime ideals of* $\mathbb{Z}$ *is the set of all subsets* $p\mathbb{Z} := \{pa \mid a \in \mathbb{Z}\}$, *where p is a prime number.*

r. An **orbifold** is a Hausdorff topological space together with an orbifold atlas. In the language of this course, we would say that an orbifold is a ringed space that is locally isomorphic to the quotient of an affine topological space under the action of a finite group - but I want to stick with the atlas definition for now.

Let $(X, \mathcal{T}_X)$ be a topological space, $U_i \in \mathcal{T}_X$, V_i be an open subset of $\mathbb{R}^n$, and Γ_i be a finite abelian group that acts faithfully and linearly on V_i. (A group action is said to be faithful if, if $g, h \in \Gamma_i$ and $g \neq h$, there exists an $x \in V_i$ such that $gx \neq hx$.) An **orbifold chart** ϕ_i is a homomorphism between V_i/Γ_i and U_i. An **orbifold atlas** of X is an open cover $\mathcal{S} \subseteq \mathcal{T}_X$ of X that is closed under finite intersection - that is, if $U_1, U_2 \in \mathcal{S}$ then $U_1 \cap U_2 \in \mathcal{S}$ - together with a finite abelian group Γ_i and an orbifold chart ϕ_i for each $U_i \in \mathcal{S}$ such that

i. If $U_i, U_j \in \mathcal{S}$ and $U_i \subseteq U_j$ then there exists an injective group homomorphism $f_{ij} : \Gamma_i \to \Gamma_j$.

ii. If $U_i, U_j \in \mathcal{S}$ and $U_i \subseteq U_j$ then there exists an open subset $W \subseteq V_j$ and a homeomorphism $\psi_{ij} : V_i \to W$ such that if $g \in \Gamma_i$ and $x \in V_i$, $\psi_{ij}(g \cdot x) = f_{ij}(g) \cdot \psi_{ij}(x)$. Such a map ψ_{ij} is called a **gluing map**.

iii. If $U_i, U_j \in \mathcal{S}$ and $U_i \subseteq U_j$, $\phi_j \circ \psi_{ij} = \phi_i$.

iv. The gluing maps are unique up to group actions - if ψ_{ij} and ψ'_{ij} are both gluing maps then there exists a $g \in \Gamma_j$ such that $g \cdot \psi_{ij}(x) = \psi'_{ij}(x)$ for all $x \in V_i$.

Example: First of all, every manifold is an orbifold when you consider the action to be that of the trivial group, every f_{ij} *to be the identity, and every gluing map to be the identity. But that's not very interesting. So instead let's consider* $X = \mathbb{R}^2/\mathbb{Z}_3$, *where* $\mathbb{Z}_3$ *is the cyclic group of three elements that acts on* $\mathbb{R}^2$ *by rotating points by* $\frac{2\pi}{3}$. *In this case, our orbifold atlas consists of a single orbifold chart - the identity map. Note that this space is not a manifold - the point* $[(0,0)] \in X$ *does not have any neighbourhood that is locally homeomorphic to* $\mathbb{R}^2$. *Open balls centered on* $[(0,0)]$ *look like "cones" - circles that have been rolled up or folded over themselves - so we call* $[(0,0)]$ *a cone point of our orbifold.*
Lukas told me that this example wasn't very interesting either, because my atlas only has one chart in it. So if you'd prefer, we can consider $X = S^2/\mathbb{Z}_3$, *where* S^2 *is the two-dimensional sphere* $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, *and* $\mathbb{Z}_3$ *acts on* S^2 *by rotating our sphere by angles of* $\frac{2\pi}{3}$ *in the x-y plane. We can give this space the following open cover:* $\mathcal{S} = \{S^2 \setminus \{(0,0,1)\}, S^2 \setminus \{(0,0,-1)\}, S^2 \setminus \{(0,0,1),(0,0,-1)\}\}$. *This set consists of the sphere with the north pole removed, the sphere wit hthe south pole removed, and the sphere wit hboth poles removed. Map each of the first two sets to* $\mathbb{R}^2/\mathbb{Z}_3$ *by stereographic projection, and map the third set to a subset of the sphere with the north pole removed. This space will look like* $\mathbb{R}^2 m \setminus \{(0,0)\}/\mathbb{Z}^3$. *The injective group homomorphism between any two sets in our open cover is just the identity. The gluing map from the set with two poles removed to the set with the north pole removed is also just the identity. The gluing map from the set with two poles removed to the set with the south pole removed is given by taking inverses:* $(x, y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$.

s. [5] There are two main conceptual ways of thinking about stacks - firstly, as 2-functors, and secondly, as categories fibered on groupoids. A **groupoid** is like a group but the pairwise operation does not need to be

[5] I got these ideas from the paper Algebraic stacks by Tomas L. Gomez, published in Proc. Indian Acad. Sci. (Math. Sci.), Vol 111, No. 1, February 2001, pp. 1 31

This paper also gave me the definitions for Fine + Coarse moduli Spaces!

as the exact definition!

defined for every pair. You can also think of a groupoid as a small category where every map is invertible. In this article we will present ~~both definitions~~, and an intuitive way of thinking about them.

~~Definitions~~

Let c and C be categories, and let $\mathcal{F} : c \to C$ be a functor. c is a **fibered category over** C if, for every $y \in c$ and every morphism $F : X \to \mathcal{F}(y)$ in C, there exists an object $F^*y \in c$ and a morphism $f : F^*y \to y$ such that $\mathcal{F}(f) = F$

Let C is a category with a Grothendieck topology and let c be a fibered category over C. c is a **prestack over** C if, if $U \in C$ and $x, y \in c$ such that $\mathcal{F}(x) = \mathcal{F}(y) = U$, the functor that takes morphisms $F : V \to U$ in C to $\mathrm{Hom}(F^*x, F^*y)$ is a sheaf

Let C is a category with a Grothendieck topology and let c be a prestack over C. c is a **stack over** C if every descent datum is effective - that is, if $U \subset c$ is open, $\{U_i\}$ is an open cover of U, X_i are objects in $\mathcal{F}(U_i)$ and $\psi_{ij} : X_j|_{ij} \to X_i|_{ij}$ are morphisms such that $\psi_{ij}|_{ijk} = \psi_{jk}|_{ijk} = \psi_{ik}|_{ijk}$, there exists an $X \in \mathcal{F}(U)$ and a set of isomorphisms $\psi_i : X|_i \to X_i$ such that $\psi_{ji} \circ \psi_i|_{ij} = \psi_j|_{ij}$. This condition means that objects can be glued together.

An **algebraic stack** is a stack in groupoids X over the etale site such that the diagonal map of X is representable and there exists a smooth surjection from (the stack associated to) a scheme to X.

Note that this definition is not complete, because I didn't define what a Grothendieck topology or an etale site is. The definition of representable is given in zection (z) when defining a fine moduli space. **Intuitively, it is helpful to think about an algebraic stack as an orbifold for which you remember information about the stabilizers of your group actions.** The example explains this better

Example: Let X be a scheme. Let G be an affine algebraic group acting on X.(An algebraic group is a group that is also an algebraic variety, such that multiplication and inversion are regular maps on the variety. An affine algebraic group is an algebraic group that is also an affine variety.) Any finite group is an affine algebraic group, so let's let $G = \mathbb{Z}/6\mathbb{Z}$, let $X = \mathbb{C}$ (equipped with the Zariski topology and sheaf of regular functions - see (d) for more details), and define the action of G on X by the following map

$$\rho : \mathbb{Z}/6\mathbb{Z} \to \mathbb{C}$$
$$[a] \mapsto e^{\frac{2\pi i a}{6}}$$

Thinking of $\mathbb{C}$ as $\mathbb{R}^2$, the generator of $\mathbb{Z}/6\mathbb{Z}$ corresponds to a rotation around the origin by an angle of $\frac{2\pi}{6}$. I have chosen this group action very specifically - the normal subgroup $H := \{[0], [3]\} \subset \mathbb{Z}/6\mathbb{Z}$ acts trivially on our scheme. So we can take $\overline{G} := G/H$ to get an affine algebraic group that acts freely on all points of our space X except the origin.
If we were to take a quotient of X by G it would be the same as taking a quotient of X by $\overline{G}$. This is because $X/G := X/\sim$, where $x \sim y$ iff there exists a $g \in G$ such that $gx = y$. But H acts trivially on X, so if there exists a $g \in G$ such that $gx = y$ then there exists a $\overline{g} \in \overline{G}$ such that $\overline{g}x = y$. In this way, we can say that the quotient scheme X/G "forgets" about H.
Suppose now that we want to define the quotient scheme but we do not want to forget about H. To do this, we define the quotient stack $[X/G]$. This is the set of all pairs (p, G_x), where $p \in X/G$ is an orbit of an element of X under the action of G, or an equivalence class of points in X under the equivalence relation defined above. Fixing a point $x \in p$, we define $G_x = \mathrm{Stab}(X) = \{g \in G \mid gx = x\}$. So, for this group action, $[X/G] = \{(p, G_x)\}$. For every $p \neq 0$, $G_x = H$, and if $p = 0, G_x = G$. Note that $[X/G]$ is now a different object to $[X/\overline{G}]$. You can visualise $[X/G]$ as the quotient space, but with a stack of two points at each point except zero, which has a stack of six points on top of it.

1) Let K be an algebraically closed field. Let $f \in K[x, y]$ be an irreducible polynomial of two variables of degree d with coefficients in K. A **curve of degree** d is the affine variety in the affine 2-space over K defined to be the set $Z(f) = \{(x, y) \mid x \in K, y \in K, f(x, y) = 0\}$.

Example: Let our curve be the polynomial $f(x, y) = x - 1$. This is clearly irreducible. Then the set of all solutions to this equation is the curve $Z(f) = \{(x, y) \in \mathbb{C}^2 \mid x = 1\}$. This forms a 1-complex dimensional subspace of our two dimensional space. From a Calculus 1 perspective, this space looks like a 2-D plane in 4-D (Real) space. For a slightly more interesting curve, let $f(x, y) = x - y$. The zero set of this polynomial also just looks like a plane. For a more interesting example, we will recycle our irreducible curve from part (f), where we claimed (without proof) that the polynomial $f(x, y) = x^2 + y^2 - 1$ was irreducible. The zero

Curve, Surface, Hypersurface definitions from Hartshorne.

Beginning Mike 661260

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set of this function also forms a curve, but I couldn't tell you what it looks like. One of the goals of algebraic geometry is to be able to classify the topology of curves like these.

u. Let K be an algebraically closed field. Let $f \in K[x, y, z]$ be an irreducible polynomial of three variables with coefficients in K. A **surface** is the affine variety in the affine 3-space over K defined to be the set $Z(f) = \{(x, y, z) \mid x \in K, y \in K, z \in K, f(x, y, z) = 0\}$.

Example: Let our curve be the polynomial $f(x, y, z) = x - 1$. This is clearly irreducible. Then the set of all solutions to this equation is the curve $Z(f) = \{(x, y, z) \in \mathbb{C}^2 \mid x = 1\}$. This forms a [illegible]-complex dimensional subspace of our three dimensional space. From a Calculus 1 perspective, this space looks like a 4-D analog of a plane in 6-D (Real) space. Sorry I couldn't come up with any more interesting examples in time!

do all surfaces lie in affine 3space?

v. Let K be an algebraically closed field. Let $n \in \mathbb{Z}$ such that $n > 3$. Let $f \in K[x_1, \ldots, x_n]$ be an irreducible polynomial of n variables with coefficients in K. A **hypersurface** is the affine variety in the affine n-space over K defined to be the set $Z(f) = \{(x_1, \ldots, x_n) \mid x_1 \in K, \ldots, x_n \in K, f(x_1, \ldots, x_n) = 0\}$.

Example: We will extend our previous example in the obvious way. Let our curve be the polynomial $f(x_1, \ldots, x_n) = x_1 - 1$. This is clearly irreducible. Then the set of all solutions to this equation is the curve $Z(f) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_1 = 1\}$. This forms a $n-1$-complex dimensional subspace of our n-dimensional space. I hope this makes the point that a hypersurface is nothing scary - it's just an affine space with one dimension removed by imposing a condition. This condition is that the points in our hypersurface must solve a specified polynomial equation.

do all hypersurfaces lie in affine space? What is a Stein manifold?

w. This will be defined in more detail in sections (z) and (y). For a formal definition, a **moduli space** is a space that is either a coarse moduli space or a fine moduli space. To avoid repeating ourselves, we will not define it yet, but we will give some intuition about it here

A moduli space is a space under an equivalence generated by some geometric condition.

Example. This example comes from Prof. Arun Ram. I bet you didn't expect a citation, did you?
Let our large space be $\mathbb{R}^2$. We will take points in this space modulo the following geometric equivalence condition: if the circle that is centered on the origin and passes through the point x is the same circle as the circle that is centered on the origin and passes through the point y, then $x \sim y$.
What can we conclude about this space? Properly defining this space is equivalent to specifying how much information you need to supply in order to uniquely define a circle centered on the origin. We know straightaway that angle doesn't matter - if $\|x\| = \|y\|$, then $x \sim y$. In fact, this information is enough to fully define our moduli space: $X := \mathbb{R}^2/\sim$, where if $\|x\| = \|y\|$, then $x \sim y$. It turns out this space is isomorphic to $\mathbb{R}_{\geq 0}$ - once we have supplied a radius $r \in \mathbb{R}_{\geq 0}$, we have defined our circle uniquely.
But why force r to be positive? Why can't we have a negative radius? This is because a circle with a radius of $-r$ is the same as a circle with radius r. Expressing that condition a litte bit more algebraically, we can say that $X \cong \mathbb{R}/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}$ in the following way: $[0]x = x$ and $[1]x = -x$ for all $x \in \mathbb{R}$.
This gives us a proper-looking moduli space: as something that looks like a scheme modulo some equivalence relation generated by a group action. If we include information about our stabilisers of elements, we get a coarse moduli space, which is a geometrical object called a stack. If we instead forget about that stuff, we get a sheaf of sets, which is called a fine moduli space.

x. This definition comes from Peter Scholze's answer on MathOverflow[6], and his arXiv overview on perfectoid spaces (1111.4914).

A **perfectoid space** is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ that is locally isomorphic to an affinoid perfectoid space.

definition?

Let R be a Banach ring. The set of **powerbound elements in R** is the set $R^\circ := \{x \in R \mid \sup(\{\|x^n\| \mid n \in \mathbb{Z}_{>0}\}) < \infty\}$.

A **perfectoid field** is a complete topological field K (of characteristic 0, with a residue field of characteristic $p > 0$) whose topology is induced by a nondiscrete valuation of rank 1, such that the Frobenius map Φ is surjective on K°/p.

let K be a perfectoid field. A **perfectoid K-algebra** is a Banach K-algebra R such that the set of powerbounded elements $R^\circ \subseteq R$ is bounded, and such that the Frobenius map Φ is surjective on R°/p.

[6] https://mathoverflow.net/questions/65729/what-are-perfectoid-spaces

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Let R be a perfectoid K-algebra, and $R^+ \subseteq R^\circ$. An **affinoid perfectoid space** is the space $\mathrm{Spa}(R, R^+)$, consisting of all continuous valuations on R that are ≤ 1 on R^+, with topology generated by its rational subsets, and a structure sheaf $\mathcal{O}_X$ that consists of functions whose absolute value is ≤ 1 everywhere.

Let A be a ring and Γ be an ordered multiplicative abelian group. A **valuation** of A is a map $v : A \to \Gamma \cup \{0\}$, such that:

i. $v(a) = 0$ if and only if $a = 0$.

ii. If $a, b \in A$ then $v(ab) = v(a) \cdot v(b)$

iii. If $a, b \in A$ then $v(a+b) \leq \max(v(a), v(b))$, and if $v(a) \neq v(b)$ then $v(a+b) = \max(v(a), v(b))$.

The valuation $v : A \to \{0, 1\}$ that sends $0 \to 0$ and every other element to 1 is called the trivial valuation. A valuation is of rank 1 if it is nontrivial and $\Gamma \subseteq \mathbb{R}^*$

To define this properly, I would need to further define the followng words: Frobenius map, rational subset, and residue field.

Example.[7] *Let $R = \mathbb{C}\langle T\rangle$, the ring of convergent power series' with coefficients in the complete algebraically closed field $\mathbb{C}$. Let $R^+ = R^\circ = \mathbb{C}^\circ\langle T\rangle$, where $\mathbb{C}^\circ = \{x \in \mathbb{C} \mid \|x\| \leq 1\}$ is the set of powerbounded elements in $\mathbb{C}$. Then $\mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space, and hence a perfectoid space.*
Let's look at some points in this perfectoid space. Firstly, for any $x \in \mathbb{C}^\circ$ we can take any valuation of the following form:

$$V_x : \mathbb{C}\langle T\rangle \to \mathbb{R}_{geq0}$$
$$\sum_{n=1}^{\infty} a_n T^n \mapsto \left\| \sum_{n=1}^{\infty} a_n x^n \right\|$$

However, this is not a complete list of continuous valuation maps that are ≤ 1 on R^+. For example, let $r \in \mathbb{R}$ such that $0 \leq r \leq 1$, and let $x \in \mathbb{C}^\circ$. Then the map:

$$V_{x,r} : \mathbb{C}\langle T\rangle \to \mathbb{R}_{geq0}$$
$$\sum_{n=1}^{\infty} a_n (T - x)^n \mapsto \sup(\|a_n\| r^n)$$

is also a point in our perfectoid space of maps.

v. **Read entry (z) first!! Knowing the definition of a fine moduli space is a prerequisite to my definition of a coarse one.**

Fix a scheme S. Let M and F be schemes over S. We say that M **corepresents** F if there exists a natural transformation of functors $\phi : F \to \mathrm{Hom}_S(-, M)$ such that, given any other scheme N over S, and any natural transformation $\psi : F \to \mathrm{Hom}_S(-, N)$ there exists a unique natural transformation $\eta : \mathrm{Hom}_S(-, M) \to \mathrm{Hom}_S(-, N)$ such that $\psi = \eta \circ \phi$. In other words, the below diagram commutes:

$$\begin{tikzcd} F \arrow[d, "\phi"] \arrow[dr, "\psi"] & \\ \mathrm{Hom}_S(-, M) \arrow[r, "\eta"] & \mathrm{Hom}_S(-, N) \end{tikzcd}$$

A **coarse moduli space** is a scheme M that corepresents F such that, if k is any algebraically closed field, the map $\phi(k) : F(\mathrm{Spec}(k)) \to \mathrm{Hom}_S(\mathrm{Spec}(k), M)$ is bijective.

See (w) for an example.

z. We have to work through a bit of theory before we can properly define a fine moduli space.

Firstly, any ring homomorphism $\phi : B \to A$ induces a morphism of ringed spaces $\phi^a : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ that maps every prime ideal $p \subseteq A$ to its preimage under ϕ: $\varphi^{-1}(p) \subseteq A$.

[7] From Peter Sholze's 2011 paper Perfectoid Spaces, available on arXiv: 1111.4914. This is Example 2.20, re-written to look more proof-mechanical.

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Fix a scheme. S. A **scheme over** S is a scheme M together with a morphism of schemes $M \to S$. A **morphism of schemes** is a continuous map $f : M \to S$ such that for any $x \in M$ there exists a pair of open sets $U \in \mathcal{T}_M$ and $V \in \mathcal{T}_M$ such that (i) $U \cong \mathrm{Spec}(A)$ and $V \cong \mathrm{Spec}(B)$ for two commutative rings A and B and $f|_U$ is induced by a ring homomorphism $\phi : B \to A$.

Let M and N be schemes over S - that is, there exists two morphisms of schemes $p : M \to S$ and $q : N \to S$. A **morphism of schemes over** S is a morphism of schemes $f : M \to N$ such that $q \circ f = p$. In other words, a morphism of schemes over S is a morphism of schemes from M to S such that the below diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "p"'] & N \arrow[d, "q"] \\ & S \end{tikzcd}$$

We can now define the functor $\mathrm{Hom}_S(-, M)$. Let B be a scheme over S. Then $\mathrm{Hom}_S(-, M)$ sends B to $\mathrm{Hom}_S(B, M)$, the set of all morphisms from B to M over S. This functor is a sheaf of sets when the category of all schemes is given the Zariski topology.

Let F be a sheaf of sets. F is **representable** if there exists a scheme M such that F is isomorphic to $\mathrm{Hom}_S(-, M)$. Such an M is called a **fine moduli space**

See (w) for an example.

NEEDS TO BE LARGER FONT

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2 Paper Review - Mori Dream Spaces and Blowups by Ana-Maria Castravet

Shigefumi Mori was born in 1953, and works as an algebraic geometer at Kyoto university. His lifelong goal has been to classify algebraic varieties - work for which he was awarded the Fields medal in 1990. This work led to the development of the Minimal Modal Program, which aims to describe projective varieties in terms of birational geometry. A **Mori dream space** is a special kind of projective variety that lends itself to this classification very nicely. Work contributing to the Minimal Modal Program earned Prof. Caucher Birkar his 2018 Fields Medal, making this research particularly relevant today.

a Fields medal does not make research relevant. ~~Relevant~~ Maybe the other implications has some validity. Maybe not. What is "relevant" research? Relevant to what?

This paper is concerned with **toric varieties**, which are algebraic varieties that are acted on by algebraic torii. An **algebraic torus** ~~is a generalization of a torus for fields other than $\mathbb{C}$ - the torus we all~~ know and love can be thought of as finite copies of the group of roots of unity of $\mathbb{C}$, and an algebraic torus is just finite copies of the multipicative group over any field. More specifically, Castravet seeks to answer the following question: When is the blowup of a (projective, $\mathbb{Q}$-factorial) toric variety not a Mori dream dpace? In other words, when does blowing up a point of a special kind of variety give you a space that is not easy to classify using birational geometry?

?? you mean copies of S^1 ?? S^1 is not "roots of unity" unless you are using some nonstandard defn. of "roots of unity"!

Castravet's paper takes the style of a review. After the introduction, the second section defines what it means to be a Mori dream space, and introduces several key properties of them, like how small $\mathbb{Q}$-factorial modulations of Mori dream spaces continue to be Mori dream spaces, and how their cone of movable divisors (whatever they are) can be contracted into a union of simpler cones, called Mori chambers.

Section three is rich with examples of Mori dream spaces. Every projective $\mathbb{Q}$-factorial toric variety is a Mori dream space, as is every projective $\mathbb{Q}$-factorial variety with a Picard number of 1. (Don't worry about what a Picard number is.) Many of the examples I gave in Part 1 of this assignment satisfy these conditions - the Riemann sphere $\mathbb{P}^1$ is a projective $\mathbb{Q}$-factorial toric variety, and so it is a Mori dream space, as is $\mathbb{P}^2 := \mathbb{C}^3 \setminus \{(0,0,0)\}/\sim$, under the equivalence relation $(a_0, a_1, a_2) \sim (\lambda a_0, \lambda a_1, \lambda a_2)$ for all $\lambda \in \mathbb{C}^3 \setminus \{0\}$. Castravet takes it even further, stating that the blow up of $\mathbb{P}^n$ at r distinct points is a Mori dream space if and only if $\frac{1}{n+1} + \frac{1}{r-n-1} > \frac{1}{2}$. I find this kind of theory - relating a problem in algebraic geometry to a simple, yet unexpected inequality - incredibly beautiful.

Section four is titled "Structure Theory", but it's mostly just a list of things that might or might not be Mori dream spaces. This section is full of examples, but light on proofs, prefering to refer the reader to dedicated papers rather than waste space and time repeating known results. In this section, many open problems are introduced.

The remainder of this paper is all about blowups. At this point I cannot postpone explaining what a blowup is any longer. My first encounter with blow-ups came during a winter course on curvature, ran by the geometric analyst Mariel Saez. She wanted to know how networks would move under curve shortening flow - but she encountered singularities at the points where different curves met. How would these points move when they are tugged along by two or more moving curves?

A blowup is a kind of geometrical hack that allows you to answer questions like this. You take a space (a variety, or a manifold of some kind), and you "cut out" the subspace you want to blow up. For Mariel's purpose and for Castravet's, this subspace is always just a point, but in general it can be any subspace with codimension ≥ 1. Once you have cut out your point (or subspace), you then replace it with (roughly) the space of all lines coming out of that point (or subspace). This allows you to replace a 0-dimensional singularity with a space that has volume, and once you have volume, you can ask how curves flow through this volume, before "blowing down"[8] to return to the space you started with. This kind of analysis is very useful for dealing with singularities. I think string theorists use it a bunch. The name blowup comes from the idea of zooming in on a singular point forever until it looks like it has an actual volume, similar to how a microscope blows up an ant until its teeth looks like swords and you can make out all of this fine structure from what previously just looked like a dot.

Castravet wants to know when the blowup of a projective $\mathbb{Q}$-factorial surface with Picard number 1 at a single point is a Mori dream space. In particular, she wants to know when the blowup of a weighted projective plane is a Mori dream space. The **weighted projective plane** of an algebraically closed field k is the projectivisation of the subspace of $k[x_1, \ldots, x_n]$, where each x_1 has a fixed degree. As far as Castravet is concerned, n always equals 3. Her weights are written as (a, b, c) (that is, x has degree a, y has degree b, and z has degree c), and the weighted projective space is written as $\mathbb{P}(a, b, c)$. This space forms a toric, projective, $\mathbb{Q}$-factorial surface with Picard number 1.

Take an arbitrary point on $\mathbb{P}(a, b, c)$. Using the torus action that this space has because it is a toric variety, we can "spin" this point around so that it corresponds to the identity e of our torus action. Then, if we want to know about the properties of the blow-up of $\mathbb{P}(a, b, c)$ at any point, we only need to look at its blow-up at e.

[8] This author believes that the opposite of a blowup should be a suckdown, not a blowdown

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Of all of the results in this paper, my favourite is the following (Theorem 6.1, proved by J. L. Gonzalez and K Karu in 2016):

Assume k is an algebraically closed field with characteristic 0.
Assume $(a, b, c) \in \mathbb{Z}^3_{>0}$. Assume there exists an $m \in \mathbb{Z}_{>0}$ such that one of the following is true:

- $m > 3, 3 \mid m$, and $(a, b, c) = (7m - 3, 5m^2 - 2m, 8m - 3)$
- $m > 4, 3 \mid 7m - 10, m \not\equiv -7 \pmod{59}$, and $(a, b, c) = (7m - 10, 5m^2 - 7m + 1, 8m - 3)$

Then the blowup of $\mathbb{P}(a, b, c)$ at e is **not** a Mori dream space.

I like this theorem, because it reminds me of Euclid's formula for generating Pythagorean triples - taking an m that satisfies certain numerical conditions, and plugging it into a set of formulas gives you a geometrical object for which a certain fact is true.

In section 7, things get a little more complicated. Castravet uses a class of spaces called the Losev-Manin spaces to prove some facts about the blowups of higher-dimensional toric varieties. A **Losev-Manin space** $\overline{LM}_n$ is the blowup of the projective space $\mathbb{P}^{n-3}$ at a set of $n-2$ carefully selected points. These Losev-Manin spaces connect to weighted projective planes in a surprising way, illustrated by Corollary 7.4:

Let a, b, c be a set of positive pairwise coprime integers. Let $n = a + b + c + 8$.
If $\mathrm{Bl}_e \overline{LM}_n$ is a Mori dream space then $\mathrm{Bl}_e \mathbb{P}(a, b, c)$ is a Mori dream space.

Note: $\mathrm{Bl}_e X$ is shorthand for the blowup of a space X at the point e.

Castravet uses this result in two ways. Firstly, this allows her to conclude facts about her weighted projective planes by looking at this higher-dimensional spaces; but she also uses it the other way. Because she knows (for example) that $\mathrm{Bl}_e \mathbb{P}(12, 13, 17)$ is not a Mori dream space, she can instantly conclude using this theorem that $\mathrm{Bl}_e \overline{LM}_{50}$ is not a Mori dream space.

The final section of this paper connects Castravet's question about Mori dream spaces to the "interpolation problem" - asking whether or not it is possible to find polynomial curves of a small enough degree that pass through a certain set of points. Working without proof, Castravet comments that every polarized toric projective surface with Picard number ρ possesses an ample (Q-factorial, Cartier) divisor that "corresponds" somehow to a rational polytope with $\rho + 2$ vertices. What she means by this correspondance is still mysterious to me. A **polytope** is the any-dimensional generalisation of a polygon or a polyhedron. Understanding this is not so important, as later on she deals exclusively with surfaces of Picard number 1, for which the polytope is simply a triangle. A polytope is **rational** when all of its vertices have coefficients in $\mathbb{Q}$. For any such a polytope, there exists an integer d such that if $(x, y) \in \Delta$ is a vertex then $(dx, dy) \in \mathbb{Z}$. We define $d\Delta := \{(dx, dy) \in \mathbb{R}^2 \mid (x, y) \in \Delta\}$.

We are now ready to state the theorem that relates the main question of Castravet's paper (When is the blowup of a toric variety not a Mori dream space?) to a question about interpolation. This is Proposition 8.6 in the paper, slightly rephrased:

Let X_Δ be a polarized toric surface with Picard number 1.
Let H be an ample, Q-factorial, Cartier divisor of X_Δ that (i) corresponds to a triangle Δ, and (ii) satisfies $w = H^2 < 1$.
Then the blowup of X_Δ at e is **not** a Mori dream space if and only if for every point $(x, y) \in \Delta$, and every $d \in \mathbb{Z}_{>0}$ such that $d\Delta$ has integer coefficients, there exists a curve of degree $dw - 1$ that passes through every point of $d \in \mathbb{Z}_{>0} \cap \mathbb{Z}^2$ except for (dx, dy).

As an example of this, Castravet translates the question of whether the blowup of $\mathbb{P}(9, 10, 13)$ at the origin is **not** a Mori dream space into the following interpolation problem. Let $\Delta \subset \mathbb{R}^2$ be the triangle with vertices at the points $(0, 0)$, $(10, 40)$, and $(36, 27)$. Let $m_q = \lfloor\sqrt{1170q}\rfloor + 1$. Is it true that for all $q \in \mathbb{Z}_{>0}$, and for any point $(x, y) \in q\Delta$ for which $x, y \in \mathbb{Z}$, there exists a curve of degree at most m_q that passes through every point of $q\Delta \cap \mathbb{Z}^2$ except for (i, j)? These two questions are *equivalent*: answering either one affirmatively instantly proves the other one to be true.

Unfortunately, it seems that this new problem is no simpler than the old one. It is also yet to be determined whether or not Proposition 8.6 works in higher dimensions. However, this result is still very useful, because it means that any further advancements in either field will help to benefit the other. If someone invents a new algorithm that helps solve these interpolation problems, we will know a lot more about Mori dream spaces and the minimal modal program, and if we succeed in classifying all blowups of Mori dream spaces, we will get lots of theorems about interpolation for free. Under this theorem, two seemingly unrelated areas of mathematics have been shown to be deeply linked. For me, it is these rare, unexpected connections that make mathematics beautiful.

nice