Sample Exam questions 3

MAST90097 Algebraic Geometry
Semester II 2018
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(1) (Harder Chapter 1 Example 1)

   (a) Carefully define a category.
   (b) Carefully define the category **Ens** of sets.
   (c) Prove that **Ens** is a category.

(2) (Harder Chapter 1 Example 2)

   (a) Carefully define a category.
   (b) Carefully define a *functor*.
   (c) Let $\mathbb{F}$ be a field. Carefully define the category $\mathbf{Vect}_k$ of vector spaces over k.
   (d) Prove that $\mathbf{Vect})_k$ is a category.
   (e) Define the *forgetful functor* from $\mathbf{Vect})_k$ to **Ens**.

(3) (Harder Chapter 1 Example 3)

   (a) Carefully define the category **Groups** of groups.
   (b) Prove that **Groups** is a category.
   (c) Let A be a ring. Carefully define the category $\mathbf{Mod}_A$ of A-modules.
   (d) Prove that $\mathbf{Mod}_A$ is a category.
   (e) Explain why the terms abelian group and vector space should be deprecated.

(4) (Harder Chapter 1 Example 4)

   (a) Carefully define the category **Top** of topological spaces.

(b) Prove that **Top** is a category.

(5) (Harder Chapter 1 Example 5)

(a) Explain how a poset $(\mathcal{I}, \leqslant)$ can be viewed as a category.

(b) Explain how a group G can be viewed as a category.

(c) Explain how a topological space $(X, \mathcal{T})$ can be viewed as a category.

(6) (a) Carefully define the category **Metric** of metric spaces.

(b) Prove that **Metric** is a category.

(7) (Harder Chapter 1 Example 6)

(a) Carefully define a functor of "taking the dual" from $\mathbf{Vect}_k$ to $\mathbf{Vect}_k$.

(b) Prove that "taking the dual" is a contravariant functor.

(8) (Harder Chapter 1 Example 7)

(a) Prove that $\mathbb{R}^m \not\cong \mathbb{R}^n$ if $m \neq n$.

(b) Describe the homology groups $H_i(\mathbb{R}^m, \mathbb{Z})$.

(9) (Harder Chapter 1 Example 8)

(a) Define *product* in the categorical sense.

(b) Define the product $X \times Y$ of two sets X and Y.

(c) Prove that the product $X \times Y$ of two sets X and Y is a product in the categorical sense.

(10) (Harder Chapter 1 Example 9)

(a) Carefully define the ring $\widehat{\mathbb{Z}}$.

(b) What is $1+1+1$ in the ring $\widehat{\mathbb{Z}}$? (with proof, of course).

(c) What is $\frac{1}{3}$ in the ring $\widehat{\mathbb{Z}}$? (with proof, of course).

(d) Carefully define the ring $\mathbb{Z}_p$.

(e) Carefully define the topology on $\mathbb{Z}_p$.

(f) Show that $\mathbb{Z}$ is a dense subring of $\mathbb{Z}_p$.

(e) Prove that $\widehat{\mathbb{Z}}$ is a product of $\mathbb{Z}_p$.

(f) Give an example of a zero divisor in $\widehat{Z}$ (with proof, of course).

(11) (Harder Chapter 1 Example 10)

(a) Prove that products exists in the category of sets.

(b) Carefully define the category of fields.

(c) Give an example (with proof, of course) of two fields $\mathbb{F}$ and $\mathbb{K}$ such that the product $\mathbb{F} \times \mathbb{K}$ does not exist (in the category of fields).

(12) (Harder Chapter 1 Example 11)

(a) Carefully define a field.

(b) Carefully define the algebraic closure $\overline{\mathbb{F}}$ of a field $\mathbb{F}$.

(c) Carefully define the Galois group $Gal(\mathbb{K}/\mathbb{F})$.

(d) Prove that the Galois group $Gal(\overline{\mathbb{F}}/\mathbb{F})$ is a projective limit of finite groups.

(13) (Harder Chapter 1 Example 12)

(a) Carefully define the *Krull topology* on a product of finite sets.

(b) Determine when the Krull topology on a product of finite sets coincides with the product topology (coming from the discrete topology on each factor).

(c) Carefully define a *profinite set.*

(d) Carefully define a *profinite group.*

(e) Carefully define a *profinite ring.*

(e) Prove that $\widehat{\mathbb{Z}}$ is a profinite ring.

(e) Prove that $Gal(\overline{\mathbb{F}}/\mathbb{F})$ is a profinite group.

(14) (Harder Chapter 1 Exercise 1)

(a) Carefully define disjoint union in the category theoretic sense.

(b) Are there disjoint unions in the category $\mathbf{Vect}_k$? If so, construct the disjoint union as a vector space. If not, prove that it does not exist.