Sample Exam questions 2

MAST90097 Algebraic Geometry
Semester II 2018
Lecturer: Arun Ram

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(1) (What is $Pic(\mathbb{P}^n)$?)

   (a) Define $\mathbb{P}^n$
   (b) Define $Pic(\mathbb{P}^n)$
   (c) Explain, with proof, how $Pic(\mathbb{P}^n)$ is an abelian group.
   (d) Prove that $Pic(\mathbb{P}^n) \cong \mathbb{Z}$.

(2) ($H^0(\mathbb{P}^n, \mathcal{O}(d))$) Correct the following questions appropriately, and answer them thoroughly, including necessary proofs.

   (a) Define $H^0(\mathbb{P}^n, \mathcal{O}(d))$.
   (b) Prove that $\dim_k(H^0(\mathbb{P}^n_k, \mathcal{O}(d)) = \binom{n+d-1}{n}$.
   (b) Prove that $\dim_k(H^0(\mathbb{P}^n_k, \mathcal{O}(d)) = 0$.
   (c) Explain how $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is a $GL_n(\mathbb{C})$-module?
   (d) Prove that $H^0(\mathbb{P}^n, \mathcal{O}(d)) \cong Sym^d(\mathbb{C}^n)$ as $GL_n(\mathbb{C})$-modules.

(2) ($\mathcal{O}(d)$) Let $d \in \mathbb{Z}$ and let $X = \mathbb{P}^n$.

   (a) Construct $\mathcal{O}(d)$.
   (b) Prove that $\mathcal{O}(d)$ is a line bundle.

(3) (a) State the gluing theorem.
   (b) Prove the gluing theorem.

(4) Carefully define $\mathcal{O}_X$-module.

(5) Show that there exists a finite open cover $\mathcal{S}$ of $\mathbb{P}^n$ and isomorphisms of ringed spaces

$$\varphi\colon (U, \mathcal{T}_U^{\mathrm{Zar}}, \mathcal{O}_U) \xrightarrow{\sim} (\mathbb{A}^n, \mathcal{T}_{\mathbb{A}^n}^{\mathrm{Zar}}, \mathcal{O}_{\mathbb{A}^n}), \qquad \text{for } U \in \mathcal{S}.$$

(6) (a) Define $\mathcal{O}_{\mathbb{P}^n}$.
(b) Let $U \subseteq \mathbb{P}^n$ be open. Define *regular function on U*.
(c) Show that $\mathcal{O}_{\mathbb{P}^n}(U) = \{\text{regular functions on } U\}$.

(7) Correct the following questions appropriately, and answer them thoroughly, including necessary proofs.

(a) Define the Zariski topology on $\mathbb{P}^n$
(b) Show that the quotient topology on $\mathbb{P}^n$ coming from $\mathbb{C}^{n+1} - \{0\}$ is the Zariski topology.
(c) Show that the quotient topology on $\mathbb{P}^n$ coming from $\mathbb{C}^{n+1} - \{0\}$ is not the Zariski topology.
(d) Show that the quotient topology on $\mathbb{P}^n$ coming from $\mathbb{C}^{n+1} - \{0\}$ is quasicompact and Hausdorff.
(e) Show that the quotient topology on $\mathbb{P}^n$ coming from $\mathbb{C}^{n+1} - \{0\}$ is quasicompact and not Hausdorff.

(8) (affine algebraic sets)

(a) Carefully define *affine algebraic set*.
(b) Classify all affine algebraic sets in $k[x]$.
(c) Define a topology on $\mathbb{A}^n_k$ by using affine algebraic sets.
(d) Show that the topology defined in (c) is a topology.

(9) (The Zariski topology)

(a) Carfully define the Zariski topology.
(b) Carefully prove that the Zariski topology is a topology

(10) Correct the following questions appropriately, and answer them thoroughly, including necessary proofs.

(a) Carefully define the Zariski topology.

(b) Explain why the Zariski topology on $k[x]$ is not Hausdorff.

(c) Explain why the topology on $\mathbb{C}$ is Hausdorff.

(d) Explain why the topology on $\mathbb{C}$ is Hausdorff.

(11) Prove the lemma on gluing of sheaves.

(12) ($\mathbb{P}^n$ is a CW-complex)

(a) Carefully define CW-complex.

(b) Prove, by construction (and with proof that your construction is correct), that $\mathbb{P}^n$ is a CW-complex with one cell each of dimensions $0, 1, \dots, n$.

(13) ($\mathbb{P}^n$)

(a) Carefully define $\mathbb{P}^n$.

(b) Carefully prove that $\mathbb{P}^n \cong \{\text{lines through origin in } k^{n+1}\}$

(14) ($\mathbb{A}^n$)

(a) Carefully define $\mathbb{A}^n_k$.

(b) Carefully define regular function on U.

(c) Prove that $\mathcal{O}_{\mathbb{A}^n}(U) = \{\text{regular functions on } U\}$ is a ring.

(d) Prove that $\mathcal{O}_{\mathbb{A}_n}(\mathbb{A}^n) = k[x_1, \dots, x_n]$.

(e) Carefully define the sheaf $\mathcal{O}_{\mathbb{A}^n}$.

(f) Prove that $\mathcal{O}_{\mathbb{A}^n}$ is a sheaf on $\mathbb{A}^n$.

(15) (a) Carefully define $\mathcal{O}_{\mathbb{A}_n}(\mathbb{A}^n)$.

(b) Prove that $\mathcal{O}_{\mathbb{A}_n}(\mathbb{A}^n) = k[x_1, \dots, x_n]$.

(c) Give an example (with proof) to shows that $\mathcal{O}_{\mathbb{A}_n}(\mathbb{A}^n)$ is not always equal to $k[x_1, \dots, x_n]$.

(16) ($\mathbb{A}^n$)

(a) Carefully define the set $\mathbb{B}^n = \text{Spec}(\mathbb{C}[x_1, \dots, x_n])$.

(b) Carefully define the set $\mathbb{A}^n_{\mathbb{C}}$ as Arik did in class.

(c) Describe the relationship between the sets $\mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{B}^n$.

(d) Carefully define the topology on $\mathbb{A}^n_{\mathbb{C}}$ as Arik did in class.

(e) Carefully define the Zariski topology on $\mathbb{B}^n_{\mathbb{C}}$.

(f) Describe the relationship between the topology on $\mathbb{A}^n_{\mathbb{C}}$ and the topology on $\mathbb{B}^n$.

(g) Carefully define the structure sheaf of $\mathbb{A}^n_{\mathbb{C}}$ as Arik did in class.

(h) Carefully define the structure sheaf of $\mathbb{B}^n_{\mathbb{C}}$.

(f) Describe the relationship between the structure sheaf of $\mathbb{A}^n_{\mathbb{C}}$ and the structure sheaf of $\mathbb{B}^n$.

(17) (a) Carefully define $\mathbb{CP}^n$, the quotient of $\mathbb{C}^n - \{0\}$ with the standard topology.

(b) Show that $\mathbb{CP}^n$ is quasicompact and Hausdroff.

(c) Define an imbedding $\mathbb{CP}^{n-1}$ into $\mathbb{CP}^n$. Show that this map is well defined and injective.

(d) Define the disc D^{2n}, and a function $f_n\colon D^{2n} \to \mathbb{CP}^n$.

(e) Show that the function $f_n\colon D^{2n} \to \mathbb{CP}^n$ is continuous and surjective.

(f) Define the sphere S^{2n-1}, and a function $g_n\colon S^{2n-1} \to \mathbb{CP}^{n-1}$.

(g) Show that the function $f_n\colon D^{2n} \to \mathbb{CP}^n$ is continuous and surjective.

(h) Carefully define $D^{2n} \sqcup_{g_k} \mathbb{CP}^{n-1}$.

(i) Prove that $D^{2n} \sqcup_{g_k} \mathbb{CP}^{n-1}$ is homeomorphic to $\mathbb{CP}^n$.

(18) (a) Carefully define $\mathbb{P}^n_{\mathbb{C}}$, the quotient of $\mathbb{C}^{n+1} - \{0\}$ with the Zariski topology.

(b) Carefully define a *projective algebraic set* and how to construct a topology using projective algebraic sets.

(c) Show that the topology coming from projective algebraic sets coincides with topology on $\mathbb{P}^n_{\mathbb{C}}$ obtained from the quotient of the Zariski topology on $\mathbb{C}^{n+1} - \{0\}$. topology.

(18) (a) Carefully define a regular function on $\mathbb{P}^n_{\mathbb{C}}$.

(c) Prove that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(U) = \{\text{regular functions on } U\}$ is a ring.

(d) Prove that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(\mathbb{A}^n) = \mathbb{C}$.

(e) Carefully define the sheaf $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$.

(f) Prove that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$ is a sheaf on $\mathbb{P}^n_{\mathbb{C}}$.

(18) (a) Carefully define a holomorphic function on $\mathbb{CP}^n$.

(c) Prove that $\mathcal{O}_{\mathbb{CP}^n}(U) = \{\text{holomorphic functions on } U\}$ is a ring.

(d) Prove that $\mathcal{O}_{\mathbb{CP}^n}(\mathbb{A}^n) = \mathbb{C}$.

(e) Carefully define the sheaf $\mathcal{O}_{\mathbb{CP}^n}$.

(f) Prove that $\mathcal{O}_{\mathbb{CP}^n}$ is a sheaf on $\mathbb{CP}^n$.