Sample Exam questions 1

MAST90097 Algebraic Geometry
Semester II 2018
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(1) Define the following terms:

  (a) ringed space
  (b) morphism of ringed spaces
  (c) locally isomorphic (as ringed spaces)
  (d) structure sheaf
  (e) topological space

(2) Define the following terms:

  (a) sheaf of rings
  (b) sheaf of abelian groups
  (c) presheaf of rings
  (d) presheaf of abelian groups
  (e) $\mathcal{T}_X$ as a category
  (f) exact contravariant functor $\mathcal{F}\colon \mathcal{T}_X \to \mathcal{A}$
  (f) contravariant functor $\mathcal{F}\colon \mathcal{T}_X \to \mathcal{A}$

(3) Define the following terms:

  (a) sheaf of $\mathcal{O}_X$-modules
  (b) locally free sheaf
  (c) coherent sheaf
  (d) vector bundle

(4) Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space and let $U \in \mathcal{T}_X$.

(a) Define the subspace topology $\mathcal{T}_U$.

(b) Define the sheaf $\mathcal{O}_U$.

(c) Show that $(U, \mathcal{T}_U, \mathcal{O}_U)$ is a ringed space.

(5) Carefully define the following ringed spaces:

(a) $(\mathbb{R}^n, \mathcal{T}^{\text{std}}, C^0)$.

(b) $(\mathbb{R}^n, \mathcal{T}^{\text{std}}, C^r)$.

(c) $(\mathbb{R}^n, \mathcal{T}^{\text{std}}, C^\infty)$.

(d) $(\mathbb{C}^n, \mathcal{T}^{\text{std}}, C^{\text{an}})$.

(6) Carefully define the following ringed spaces:

(a) affine space $\mathbb{A}^n$

(b) projective space $\mathbb{P}^n$

(7) Let $(X, \mathcal{T}_X)$ be a topological space. Let $\mathcal{S}$ be an open cover of X such that if $U, V \in \mathcal{S}$ then $U \cap V \in \mathcal{S}$. Assume given

(A) For each $U \in \mathcal{S}$ a ring Γ_U,

(B) For each $U, V \in \mathcal{S}$ such that $U \cap V \neq \emptyset$ a ring isomorphism

$$g_{UV} \colon \Gamma_{U \cap V} \to \Gamma_{V \cap U}.$$

Show that there is a unique sheaf $\mathcal{O}_X$ on X such that

$$\text{if } U \in \mathcal{S} \quad \text{then} \quad \mathcal{O}_X(U) = \Gamma_U.$$

(8) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

Condition (A):

if $p \in X$ then there exists $U \in \mathcal{T}_X$ and $n \in \mathbb{Z}_{>0}$
such that $p \in U$ and $\mathcal{F}(U) \cong \mathcal{O}_X(U)^{\oplus n}$.

Condition (B):

there exists $n \in \mathbb{Z}_{>0}$ such that if $p \in X$ then there exists $U \in \mathcal{T}_X$
such that $p \in U$ and $\mathcal{F}(U) \cong \mathcal{O}_X(U)^{\oplus n}$.

Show that $\mathcal{F}$ satistifes (A) if and only if $\mathcal{F}$ satisfies (B).