Metric and Hilbert spaces Lecture 9

Properties of the real numbers

Algebraic properties

(1) If $a, b, c \in \mathbb{R}$ then $(a+b)+c = a+(b+c)$

(2) If $a, b \in \mathbb{R}$ then $a+b = b+a$.

(3) There exists $0 \in \mathbb{R}$ such that

if $a \in \mathbb{R}$ then $0+a = a$ and $a+0 = a$

(4) If $a \in \mathbb{R}$ then there exists $-a \in \mathbb{R}$

such that $a+(-a) = 0$ and $(-a)+a = 0$.

(5) If $a, b, c \in \mathbb{R}$ then $(ab)c = a(bc)$

(6) If $a, b, c \in \mathbb{R}$ then

$(a+b)c = ac+bc$ and $c(a+b) = ca+cb$.

(7) There exists $1 \in \mathbb{R}$ such that

if $a \in \mathbb{R}$ then $1 \cdot a = a$ and $a \cdot 1 = a$.

(8) If $a \in \mathbb{R}$ and $a \neq 0$ then there exists

$a^{-1} \in \mathbb{R}$ such that

$aa^{-1} = 1$ and $a^{-1}a = 1$.

(9) If $a, b \in \mathbb{R}$ then $ab = ba$.

Definition of real numbers, polynomials, p-adic

$\mathbb{R}$ is the set of decimal expansions

$$\mathbb{R} = \left\{ \pm\left(a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-\ell+1}\left(\tfrac{1}{10}\right)^{-\ell+1} + \cdots \;\middle|\; \begin{matrix} \ell \in \mathbb{Z} \\ a_i \in \{0,1,\ldots,9\} \end{matrix} \right.\right\}$$

with

$$x = y \text{ if } \lim_{k\to\infty} (x_{\le k} - y_{\le k}) = 0$$

where $x_{\le k} = x_{-\ell}\, x_{-\ell+1} \cdots x_{-1} x_0 . x_1 x_2 \cdots x_k$

$$= x_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + \cdots + x_0\left(\tfrac{1}{10}\right)^0 + x_1 \tfrac{1}{10} + \cdots + x_k\left(\tfrac{1}{10}\right)^k$$

is x up to the k^{th} decimal place.

Note: $0.999\ldots = 9\tfrac{1}{10} + 9\cdot\left(\tfrac{1}{10}\right)^2 + \cdots$

$$= 9\cdot\left(\tfrac{1}{10}\right)\left(1 + \tfrac{1}{10} + \left(\tfrac{1}{10}\right)^2 + \cdots\right)$$

$$= 9\cdot\frac{1}{10}\cdot\frac{1}{1-\frac{1}{10}} = \frac{9/10}{9/10} = 1.000\ldots$$

<u>Addition</u> and <u>multiplication</u> are defined by

$$\begin{matrix} \mathbb{R}\times\mathbb{R} \to \mathbb{R} \\ (x,y) \mapsto x+y \end{matrix} \quad\text{and}\quad \begin{matrix} \mathbb{R}\times\mathbb{R} \to \mathbb{R} \\ (x,y) \mapsto xy \end{matrix}$$

given by

$$\lim_{k\to\infty} \left((x+y)_{\le k} - (x_{\le k} + y_{\le k})\right) = 0 \quad \text{and}$$

$$\lim_{k\to\infty} \left((xy)_{\le k} - x_{\le k}\cdot y_{\le k}\right) = 0$$

where $\lim_{k\to\infty}(x_{\le k}-y_{\le k})=0$ means

if $\varepsilon\in\mathbb{Q}_{>0}$ then there exists $\ell\in\mathbb{Z}_{>0}$ such that if $k\in\mathbb{Z}_{>0}$ then $|x_{\le k}-y_{\le k}|<\varepsilon$.

Polynomials

$$\mathbb{R}((t))=\left\{a_{-\ell}t^{-\ell}+a_{-\ell+1}t^{-\ell+1}+\cdots \;\middle|\; \begin{matrix}\ell\in\mathbb{Z}\\ a_i\in\mathbb{R}\end{matrix}\right\}$$

$$\cup|$$

$$\mathbb{R}[[t]]=\{a_0+a_1t+a_2t^2+\cdots \mid a_i\in\mathbb{R}\}$$

$$\cup|$$

$$\mathbb{R}[t]=\left\{a_0+a_1t+a_2t^2+\cdots \;\middle|\; \begin{matrix}a_i\in\mathbb{R}\text{ and all but}\\ \text{a finite number of}\\ a_i\text{ are }0\end{matrix}\right\}$$

7-adic numbers

$$\mathbb{Q}_7=\left\{a_{-\ell}7^{-\ell}+a_{-\ell+1}7^{-\ell+1}+\cdots \;\middle|\; \begin{matrix}\ell\in\mathbb{Z}\\ a_i\in\{0,1,\ldots,6\}\end{matrix}\right\}$$

$$\cup|$$

$$\mathbb{Z}_7=\{a_0+a_17+a_27^2+\cdots \mid a_i\in\{0,1,\ldots,6\}\}$$

$$\cup|$$

$$\mathbb{Z}=\left\{a_0+a_17+a_27^2+\cdots \;\middle|\; \begin{matrix}a_i\in\{0,1,\ldots,6\}\text{ and}\\ \text{all but a finite number}\\ \text{of }a_i\text{ are }0\end{matrix}\right\}$$

Real numbers

$$\mathbb{R}_{\geq 0} = \left\{ a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-\ell+1}\left(\tfrac{1}{10}\right)^{-\ell+1} + \cdots \;\middle|\; a_i \in \{0,1,\ldots,9\} \right\}$$

$$\cup\!|$$

$$\mathbb{R}^{\#}_{[0,1]} = \left\{ a_0 + a_1\left(\tfrac{1}{10}\right) + a_2\left(\tfrac{1}{10}\right)^2 + \cdots \;\middle|\; a_i \in \{0,1,\ldots 9\} \right\}$$

$$\cup\!|$$

$$\mathbb{Q}^{sm}_{[0,1)} = \left\{ a_0 + a_1\left(\tfrac{1}{10}\right) + a_2\left(\tfrac{1}{10}\right)^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in \{0,1,\ldots,9\} \\ \text{with all but a finite} \\ \text{number of } a_i \text{ equal } 0 \end{array} \right\}$$

Metrics

For $a \in \mathbb{R}((t))$ define

$$\mathrm{val}_t(a) = \ell \text{ if } a = a_\ell t^\ell + a_{\ell+1} t^{\ell+1} + \cdots$$
with $a_\ell \neq 0$.

For $a \in \mathbb{Q}_7$ define

$$\mathrm{val}_7(a) = \ell \text{ if } a = a_\ell 7^\ell + a_{\ell+1} 7^{\ell+1} + \cdots$$
with $a_\ell \neq 0$.

For $a \in \mathbb{R}_{\geq 0}$ define ....

Define metrics on $\mathbb{R}((t))$ and $\mathbb{Q}_7$ and $\mathbb{R}_{\geq 0}$ by

$$d_t(x,y) = e^{-\mathrm{val}_t(x-y)}, \quad \text{for } x,y \in \mathbb{R}((t))$$

$$d_7(x,y) = e^{-\mathrm{val}_7(x-y)}, \quad \text{for } x,y \in \mathbb{Q}_7$$

$$d(x,y) = |x-y|, \quad \text{for } x,y \in \mathbb{R}.$$

Then $\mathbb{R}((t))$, $\mathbb{Q}_7$ and $\mathbb{R}$ are topological spaces.