Metric and Hilbert Spaces: Lecture 8, 10.08.2017 A. Ram ①

Limits in topological spaces

Let $(X, \mathcal{T})$ be a topological space.

Let $(x_1, x_2, \ldots)$ be a sequence in X.

A <u>limit point</u> of $(x_1, x_2, \ldots)$ is $z \in X$ such that

if $N \in \mathcal{N}(z)$ then N contains all but a finite number of points of $(x_1, x_2, \ldots)$

i.e. if $N \in \mathcal{N}(x)$ then there exists $l \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>l}$ then $x_n \in N$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
Let $f: X \to Y$ be a function. Let $a \in X$ and $z \in Y$.

Write

$$z = \lim_{x \to a} f(x) \text{ if } f \text{ satisifies}$$

if $N \in \mathcal{N}(z)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$

<u>Proposition</u> f is continuous at a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Proof $\Rightarrow$: Assume f is continuous at a.

To show: $\lim_{x\to a} f(x) = f(a)$

To show: If $N \in \mathcal{N}(f(a))$ then there exists
$P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.

Assume $N \in \mathcal{N}(f(a))$.

To show: there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.

Since f is continuous at a then $f^{-1}(N) \in \mathcal{N}(a)$.

Let $P = f^{-1}(N)$.

Then $f(P) \subseteq N$, since $f^{-1}(N) = \{x \in X \mid f(x) \in N\}$.

So $\lim_{x\to a} f(x) = f(a)$.

$\Leftarrow$: Assume $\lim_{x\to a} f(x) = f(a)$

To show: f is continuous at a.

To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.

Assume $N \in \mathcal{N}(f(a))$.

To show: $f^{-1}(N) \in \mathcal{N}(a)$.

Since $\lim_{x\to a} f(x) = f(a)$ then there exists $P \in \mathcal{N}(a)$
such that $N \supseteq f(P)$.

So $P \subseteq f^{-1}(N)$

Since $P \in \mathcal{N}(a)$ then $f^{-1}(N) \in \mathcal{N}(a)$.
So f is continuous at a //.

<u>Proposition</u> Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces. Let $\mathcal{T}_X$ be the uniform space topology on X and let $\mathcal{T}_Y$ be the uniform space topology on Y. Let $f: X \to Y$ be a function. Let $a \in X$ and $y \in Y$. Then

$\lim_{x \to a} f(x) = y$ if and only if

f satisfies:

if $E \in \mathcal{X}_Y$ then there exists $D \in \mathcal{X}_X$ such that if $x \in X$ and $(x, a) \in D$ then $(f(x), y) \in E$.

<u>Proof</u>: By definition, $\lim_{x \to a} f(x) = y$ if and only if f satisfies: If $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \ni f(P)$.

By definition of the uniform space topology, $N \in \mathcal{N}(y)$ if and only if there exists $E \in \mathcal{X}_Y$ such that $B_E(y) \subseteq N$.

Thus, $\lim_{x \to a} f(x) = y$ if and only if f satisfies:

If $B_E(y)$ is an E-neighborhood of y then there exists $B_D(a)$, a D-neighborhood at x, such that $B_E(y) \ni f(B_D(a))$.

By definition, $B_D(a) = \{x \in X \mid (x,a) \in D\}$.

So $\lim_{x\to a} f(x) = y$ if and only if f satisfies:

if $E \in \mathcal{X}_Y$ then there exists $D \in \mathcal{X}_X$ such that if $x \in X$ and $(x,a) \in D$ then $(f(x), y) \in E$. //

<u>Proposition</u> Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{T}_X$ be the metric space topology on X and let $\mathcal{T}_Y$ be the metric space topology on Y. Let $f: X \to Y$ be a function. Let $a \in X$ and $y \in Y$. Then

$\lim_{x\to a} f(x) = y$ if and only if

f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $d_X(x,a) < \delta$ then $d_Y(f(x), y) < \varepsilon$.

<u>Proof</u> By definition, $\lim_{x\to a} f(x) = y$ if and only if f satisfies: If $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$

By definition of the metric space topology,

$N \in \mathcal{N}(y)$ if and only if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(y) \subseteq N$.

Thus, $\lim_{x\to a} f(x) = y$ if and only if f satisfies:

If $B_\varepsilon(y)$ is an open ball at y then there exists $B_\delta(a)$, an open ball at x such that $B_\varepsilon(y) \supseteq f(B_\delta(a))$.

By definition, $B_\delta(a) = \{x \in X \mid d(x,a) < \delta\}$.

Thus, $\lim_{x\to a} f(x) = y$ if and only if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $d_X(x,a) < \delta$ then $d_Y(f(x), y) < \varepsilon$. //