Metric and Hilbert Spaces Lecture 7 08/08/2017 A. Ram (1)

<u>Proposition</u> Let $(a_1, a_2, \ldots)$ be a sequence in X. If $z \in X$ is a limit point of $(a_1, a_2, \ldots)$ then z is a cluster point of $(a_1, a_2, \ldots)$.

<u>Proof</u> To show: There exists a subsequence $(a_{n_1}, a_{n_2}, \ldots)$ of $(a_1, a_2, \ldots)$ which converges to z.

Let $n_1 = 1$, $n_2 = 2$, ...

Then $(a_{n_1}, a_{n_2}, \ldots) = (a_1, a_2, \ldots)$ is a subsequence of $(a_1, a_2, \ldots)$ that converges to z.

So z is a cluster point of $(a_1, a_2, \ldots)$. //

<u>Proposition</u> Let $(a_1, a_2, \ldots)$ be a Cauchy sequence in X. If $z \in X$ is a cluster point of $(a_1, a_2, \ldots)$ then z is a limit point of $(a_1, a_2, \ldots)$.

<u>Proof</u> Assume z is a cluster point of $(a_1, a_2, \ldots)$.

Assume $(a_1, a_2, \ldots)$ is Cauchy.

To show: $z = \lim_{n \to \infty} a_n$

We know there is a subsequence such that $(a_{n_1}, a_{n_2}, \ldots)$ such that $z = \lim_{k \to \infty} a_{n_k}$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_n, z) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Let $s \in \mathbb{Z}_{>0}$ such that if $r \in \mathbb{Z}_{\geq s}$ then $d(a_{n_r}, z) < \frac{\varepsilon}{2}$.

Let $c \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq c}$ then $d(a_m, a_n) < \frac{\varepsilon}{2}$.

Let $\ell = n_r$ where $r \in \mathbb{Z}_{\geq s}$ such that $n_r \geq c$.

To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_n, z) < \varepsilon$.

Assume $n \in \mathbb{Z}_{\geq \ell}$.

Then $n \geq \ell = n_r \geq c$.

To show: $d(a_n, z) < \varepsilon$.

$$
\begin{aligned}
d(a_n, z) &\leq d(a_n, a_{n_r}) + d(a_{n_r}, z) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

So $\lim_{n \to \infty} a_n = z$.

So z is a limit point of $(a_1, a_2, \ldots)$. //

<u>Proposition</u> Let (X,d) be a metric space. Let $A \subseteq X$. If A is sequentially compact then A is Cauchy compact.

<u>Proof</u> Assume A is sequentially compact.

To show: A is Cauchy compact.

To show: If $(a_1, a_2, \ldots)$ is a Cauchy sequence then there exists $z \in A$ such that z is a limit point of $(a_1, a_2, \ldots)$.

Assume $(a_1, a_2, \ldots)$ is a Cauchy sequence in A.

Since A is sequentially compact there exists $z \in A$ such that z is a cluster point of $(a_1, a_2, \ldots)$.

Since $(a_1, a_2, \ldots)$ is a Cauchy sequence then z is a limit point of $(a_1, a_2, \ldots)$.

So A is Cauchy compact. //

<u>Proposition</u> If $(a_1, a_2, \ldots)$ is a convergent sequence then $(a_1, a_2, \ldots)$ is a Cauchy sequence.

<u>Proof</u> Assume $(a_1, a_2, \ldots)$ is a convergent sequence.

So there exists $z \in X$ such that $\lim_{n \to \infty} a_n = z$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq \ell}$ then $d(a_m, a_n) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq \ell}$ then $d(a_m, a_n) < \varepsilon$.

Let $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_n, z) < \frac{\varepsilon}{2}$.

To show: If $m, n \in \mathbb{Z}_{\geq \ell}$ then $d(a_m, a_n) < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{\geq \ell}$.

To show: $d(a_m, a_n) < \varepsilon$.

$$\begin{aligned} d(a_m, a_n) &\leq d(a_m, z) + d(z, a_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So $(a_1, a_2, \ldots)$ is a Cauchy sequence. //

<u>Proposition</u> Let (X,d) be a metric space. Let $A \subseteq X$. If A is Cauchy compact then A is closed.

<u>Proof</u> Assume A is Cauchy compact.

To show: A is closed.

To show: If $(a_1, a_2, \ldots)$ is a sequence in A and $z \in X$ and $\lim_{n\to\infty} a_n = z$ then $z \in A$.

Assume $(a_1, a_2, \ldots)$ is a sequence in A and $z \in X$ and $\lim_{n\to\infty} a_n = z$.

To show: $z \in A$.

Since $(a_1, a_2, \ldots)$ is a convergent sequence in X then $(a_1, a_2, \ldots)$ is a Cauchy sequence.

Since A is Cauchy compact then there exists $w \in A$ such that $\lim_{n\to\infty} a_n = w$.

Since limits are unique in metric spaces then $z = w$.

So $z \in A$.

So A is closed. //.