Metric and Hilbert Spaces: Lecture 35 — 19.10.2017 A. Ram, UniMelb. (1)

Filters

Let X be a set. A *filter* on X is a collection $\mathcal{F}$ of subsets of X such that

(a) $\emptyset \notin \mathcal{F}$

(b) (upper ideal) If $N \in \mathcal{F}$ and $P \subseteq X$ and $P \supseteq N$ then $P \in \mathcal{F}$

(c) (closed under finite intersections) If $\ell \in \mathbb{Z}_{>0}$ and $N_1, N_2, \ldots, N_\ell \in \mathcal{F}$ then

$$N_1 \cap N_2 \cap \cdots \cap N_\ell \in \mathcal{F}$$

Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. The *neighborhood filter* of x is

$$\mathcal{N}(x) = \left\{ N \subseteq X \;\middle|\; \begin{array}{l}\text{there exists } U \in \mathcal{T} \text{ with} \\ x \in U \text{ and } N \supseteq U\end{array} \right\}$$

Let $\mathcal{F}$ be a filter on X. A *limit point* of $\mathcal{F}$ is $z \in X$ such that $\mathcal{F} \supseteq \mathcal{N}(z)$.

A *cluster point* of $\mathcal{F}$ is $z \in X$ such that there exists a filter $\mathcal{G}$ on X with

$\mathcal{G} \supseteq \mathcal{F}$ and z is a limit point of $\mathcal{G}$.

Hausdorff spaces: The Theorem is Unitdelb

limit unique $\Leftrightarrow$ Hausdorff $\Leftrightarrow$ separated $\Leftrightarrow$ neighborhood pinpointed

Let $(X, \mathcal{T})$ be a topological space.

$(X,\mathcal{T})$ is *limit unique* if $(X,\mathcal{T})$ satisfies

if $\mathcal{F}$ is a filter on X then $\mathcal{F}$ has at most one limit point.

$(X,\mathcal{T})$ is *Hausdorff* if $(X,\mathcal{T})$ satisfies

if $x, y \in X$ and $x \neq y$ then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ with $U \cap V = \emptyset$

$(X,\mathcal{T})$ is *separated* if

$\Delta(X) = \{(x,x) \mid x \in X\}$ is closed in $X \times X$, with the product topology on $X \times X$.

$(X,\mathcal{T})$ is *neighborhood pinpointed* if $(X,\mathcal{T})$ satisfies

if $x \in X$ then $\displaystyle\bigcap_{N \in \mathcal{N}(x)} \overline{N} = \{x\}$,

where $\overline{N}$ is the closure of N.

Proposition

(a) Let $(X, \mathcal{E})$ be a uniform space and let $\mathcal{T}$ be the uniform space topology on X.
Then $(X, \mathcal{T})$ is Hausdorff if and only if $\bigcap_{E \in \mathcal{E}} E = \Delta(X)$.

(b) Let (X, d) be a metric space and let $\mathcal{T}$ be the metric space topology on X.
Then $(X, \mathcal{T})$ is Hausdorff.

Compact spaces

Let X be a set. In English,
an ultrafilter on X is a maximal filter.
In math,
an ultrafilter on X is a filter $\mathcal{F}$ on X such that if $\mathcal{G}$ is a filter on X and $\mathcal{G} \supseteq \mathcal{F}$ then $\mathcal{G} = \mathcal{F}$.

The <u>Theorem</u> is

filter compact $\Leftrightarrow$ ultrafilter compact $\Leftrightarrow$ exclusion compact $\Leftrightarrow$ cover compact

Let $(X, \mathcal{T})$ be a topological space.

- $(X, \mathcal{T})$ is <u>filter compact</u> if every filter on X has a cluster point in X.
- $(X, \mathcal{T})$ is <u>ultrafilter compact</u> if every ultrafilter on X has a limit point in X
- $(X, \mathcal{T})$ is <u>exclusion compact</u> if every closed exclusion contains a finite exclusion.

In math, $(X, \mathcal{T})$ is <u>exclusion compact</u> if $(X, \mathcal{T})$ satisfies:

if $\mathcal{C}$ is a collection of closed sets with $\bigcap_{K\in\mathcal{C}} K = \emptyset$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $K_1, \ldots, K_\ell \in \mathcal{C}$ such that $K_1 \cap \cdots \cap K_\ell = \emptyset$.

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- $(X, \mathcal{T})$ is <u>cover compact</u> if every open cover has a finite subcover:

In Math, $(X, \mathcal{T})$ is <u>cover compact</u> if $(X, \mathcal{T})$ satisfies:

If $\mathcal{S}$ is a collection of open sets with $\bigcup_{U \in \mathcal{S}} U = X$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_\ell \in \mathcal{S}$ such that $U_1 \cup \cdots \cup U_\ell = X$.