

Metric and Hilbert Spaces: Lecture 34  
Comparison of  $\mathbb{R}$ ,  $\mathbb{Q}_{10}$ ,  $\mathbb{Q}_5$  and  $\mathbb{Q}_2$

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Unit 16.1b (1)

Step 1  $\mathbb{Z}_{70}$

(a)  $\mathbb{Z}_{70} \subseteq \mathbb{R}$        $3726 = 3 \cdot \left(\frac{1}{10}\right)^{-3} + 7 \cdot \left(\frac{1}{10}\right)^{-2} + 2 \cdot \left(\frac{1}{10}\right)^{-1} + 6 \cdot \left(\frac{1}{10}\right)^0$

(b)  $\mathbb{Z}_{70} \subseteq \mathbb{Q}_{10}$        $6.273 = 6 \cdot 10^0 + 2 \cdot 10^1 + 7 \cdot 10^2 + 3 \cdot 10^3$

(c)  $\mathbb{Z}_{70} \subseteq \mathbb{Q}_5$        $3.402 = 3 + 4 \cdot 5^1 + 0 \cdot 5^2 + 2 \cdot 5^3$

(d)  $\mathbb{Z}_{70} \subseteq \mathbb{Q}_2$        $1.1011 = 1 + 1 \cdot 2 + 0 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^4$

Step 2: Addition

(a) in  $\mathbb{R}$        $\begin{array}{r} 53726 \\ 67241 \\ \hline 120967 \end{array}$

(b) in  $\mathbb{Q}_{10}$        $\begin{array}{r} 6.2735 \\ 1.4276 \\ \hline 7.69021 \end{array}$

(c) in  $\mathbb{Q}_5$        $\begin{array}{r} 3.402 \\ 3.143 \\ \hline 1.1011 \end{array}$

(d) in  $\mathbb{Q}_2$        $\begin{array}{r} 1.011 \\ 0.111 \\ \hline 1.1011 \end{array}$

Step 3: The element -1

(d) in  $\mathbb{Q}_2$ : Since  $\begin{array}{r} 1.00\dots \\ 1.11\dots \\ \hline 0.00\dots \end{array}$  then  $-1 = 1.11\dots$

(c) in  $\mathbb{Q}_5$ : Since  $\begin{array}{r} 1.000\dots \\ 4.444\dots \\ \hline 0.000\dots \end{array}$  then  $-1 = 4.44\dots$

b) in  $\mathbb{Q}_{10}$ : Since  $1.000\dots$   
 $\frac{9.99\dots}{0.00\dots}$  then  $-1 \equiv 9.99\dots$

(a) in  $\mathbb{R}$ :  $-1$  must be forcefully added since

$$\begin{array}{r} a_2 \dots a_2 a_1 a_0 a_{-1} a_{-2} \dots \\ \hline 1.00\dots \\ \hline 0.00\dots \end{array}$$

has no solution.

Step 4: Multiplication

(a) in  $\mathbb{R}$ :

$$\begin{array}{r} 3726. \\ 3402. \\ \hline 7452. \\ 0000 \\ 14004 \\ 8178 \\ \hline 9585852 \end{array}$$

(b) in  $\mathbb{Q}_{10}$ :

$$\begin{array}{r} 6.273 \\ 2.043 \\ \hline 2.547 \\ .0000 \\ 40041 \\ 8718 \\ \hline 2.585859 \end{array}$$

(c) in  $\mathbb{Q}_5$ :

$$\begin{array}{r} 3.402 \\ 3.143 \\ \hline 4.3211 \\ .3402 \\ 2.338 \\ 43211 \\ \hline 4.1440131 \end{array}$$

(d) in  $\mathbb{Q}_2$ :

$$\begin{array}{r} 1.011 \\ 0.111 \\ \hline 0.000 \\ 1011 \\ 1011 \\ 1011 \\ \hline 0.1101101 \end{array}$$

Step 6: Some fractions

In  $\mathbb{Q}_5$ :  $-\frac{1}{4} = \frac{1}{1-5} = 1 + 1 \cdot 5 + 1 \cdot 5^2 + \dots = 1.1111\dots$

$-\frac{1}{2} = -\frac{1}{4} + \frac{1}{4} = 2.2222\dots$

$-\frac{3}{4} = -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3.3333\dots$

$-1 = -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 4.4444\dots$

$\frac{1}{4} = 4.333\dots$  since 
$$\begin{array}{r} 4.3333\dots \\ + 1.1111\dots \\ \hline 0.0000\dots \end{array}$$
 is  $\frac{1}{4} + \frac{1}{4} = 0$ .

$\frac{1}{2} = 3.222\dots$  since 
$$\begin{array}{r} 3.222\dots \\ + 2.222\dots \\ \hline 0.000\dots \end{array}$$
 is  $-\frac{1}{2} + \frac{1}{2} = 0$ .

$\frac{1}{3} = 2.3131\dots$  since 
$$\begin{array}{r} 2.3131\dots \\ 2.3131\dots \\ 2.3131\dots \\ \hline 1.0000\dots \end{array}$$
 is  $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ .

$\frac{1}{5} = 5^{-1} = 10.00\dots$

$-\frac{1}{5} = \frac{1}{5}(-1) = 5^{-1} \cdot (4.44\dots) = 44.444\dots$

$\frac{1}{8} = 2.4141\dots$  since 
$$\begin{array}{r} 2.414141\dots \\ 2.414141\dots \\ \hline 4.333333\dots \end{array}$$
 is  $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

Note

$$\begin{array}{r}
 2.414141 \\
 3.100000 \\
 \hline
 1.303030\dots \\
 .241414\dots \\
 \hline
 1.000000\dots
 \end{array}$$

is  $\frac{1}{8} \cdot 8 = 1.$

or we can use long division

$$\begin{array}{r}
 2.414\dots \\
 3.1 \overline{) 1.000000\dots} \\
 \underline{1.300000\dots} \\
 244444\dots \\
 \underline{211000\dots} \\
 33444\dots \\
 \underline{31000\dots} \\
 2444\dots \\
 \underline{2100\dots} \\
 334\dots \\
 \underline{310\dots}
 \end{array}$$

is  $\frac{1}{8} = 2.414\dots$

in  $\mathbb{Q}_2$

$\frac{1}{8} = 2^{-3} = 1000.00\dots$

in  $\mathbb{R}$

$\frac{1}{8} = 0.125$  since

$$\begin{array}{r}
 .125 \\
 8 \overline{) 1.0000} \\
 \underline{800.} \\
 2000\dots \\
 \underline{1600\dots} \\
 400\dots \\
 \underline{400\dots} \\
 000\dots
 \end{array}$$

in  $\mathbb{Q}_{10}$ :  $\frac{1}{8} = 5210$  and

$$\frac{1}{3} = 7.666\dots \text{ since } 7.666\dots$$

$$\frac{3.000\dots}{1.000\dots} \text{ is } \frac{1}{3} \cdot 3 = 1$$

and  $7.666\dots$

$$+ 7.666\dots$$

$$\hline 4.333\dots$$

$$+ 7.666\dots$$

$$\hline 1.000\dots$$

$$\text{is } \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$