Metric & Hilbert Spaces Ass 2, Question 1 Ass2 11a) ①

Solutions, 23 Oct. 2015

(1) Compute $\|W\|$ and an eigenvector of W with eigenvalue $\|W\|$.

The matrix of $W: \mathbb{C}^5 \to \mathbb{C}^5$ in the standard basis is

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 & 9 \\ 3 & 7 & 10 & 11 & 12 \\ 4 & 8 & 11 & 13 & 14 \\ 5 & 9 & 12 & 14 & 15 \end{pmatrix}.$$

Write $\|W\| = \|B\|$ for convienience

Then $\|B\| = \sup\left\{ \frac{\|Bu\|}{\|u\|} \mid u \in \mathbb{C}^5 \right\} = \{\|Bu\| \mid \|u\| = 1\}$

and, since $B = \overline{B}^t$, B is self adjoint and

$$\|B\| = \sup\{ |\langle Bu, u\rangle| \mid \|u\| = 1\}$$

$$= \sup\left\{ \frac{|\langle Bu, u\rangle|}{\|u\|^2} \mid u \in \mathbb{C}^5 \right\}.$$

If $u_1 = (1,1,1,1,1)$ then $\|u_1\| = \sqrt{1^2+1^2+1^2+1^2+1^2} = \sqrt{5}$

and

$Bu_1 = (15, 32, 43, 50, 55)$ so that

$\langle Bu_1, u_1\rangle = 47 + 43 + 105 = 195$ and $\frac{\langle Bu_1, u_1\rangle}{\sqrt{5}} = \frac{195}{5} = 39$

So $\|W\| = \|B\| \geq 39$

Note that $\frac{1}{15} Bu_1 \approx (1, 2, 3, 3, 3)$

Let $u_2 = (1, 2, 3, 3, 3)$ and $\|u_2\|^2 = 1^2 + 4 + 9 + 9 + 9 = 32$

Then

$$Bu_2 = (1+4+9+12+15,\ 2+12+21+24+27,\ 3+14+30+33+36,\ 4+16+33+39+42,\ 5+18+36+42+45)$$

$$= (41, 85, 116, 134, 146)$$

and $$\frac{\langle Bu_2, u_2\rangle}{\|u_2\|^2} = \frac{41+170+348+402+438}{32}$$

$$= \frac{1399}{32} = 40 + \frac{119}{32} = 40 + 3 + \frac{23}{32} \approx 43.6.$$

So $\|B\| = \|W\| \geq 43.6$.

~~Since~~ Since B is self adjoint there is a matrix K with $KK^* = 1$ and

$$KBK^{-1} = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ 0 & & & & \lambda_5 \end{pmatrix}$$ and if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$ then $|\lambda_1| = \|B\| = \|KBK^{-1}\|$

Then

$$\frac{1}{\|B\|} KBK^{-1} = \frac{1}{\lambda_1} KBK^{-1} = \begin{pmatrix} \lambda_1/\lambda_1 & & & & 0 \\ & \lambda_2/\lambda_1 & & & \\ & & \lambda_3/\lambda_1 & & \\ & & & \lambda_4/\lambda_1 & \\ 0 & & & & \lambda_5/\lambda_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & 0 \\ & a_2 & & & \\ & & a_3 & & \\ & & & a_4 & \\ 0 & & & & a_5 \end{pmatrix}, \quad \text{with } a_2 < 1,\ a_3 < 1,\ a_4 < 1,\ a_5 < 1.$$

So $KB^2K^{-1} = \begin{pmatrix} 1 & & & & \\ & a_2^2 & & & \\ & & a_3^2 & & \\ & & & a_4^2 & \\ & & & & a_5^2 \end{pmatrix}$ and $KB^3K^{-1} = \begin{pmatrix} 1 & & & & 0 \\ & a_2^3 & & & \\ & & a_3^3 & & \\ & & & a_4^3 & \\ 0 & & & & a_5^3 \end{pmatrix}$

and $\lim_{k\to\infty} KB^kK^{-1}$'s $\begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$.

So, if $u \in \mathbb{C}^5$, then

$u_1 = Bu$, $u_2 = B^2u$, $u_3 = B^3u, \ldots$

is a sequence of vectors and

$\lim_{k\to\infty} u_k = \lim_{k\to\infty} B^k u$ is an eigenvector of B.

So $u_1 = Bu$, $u_2 = B^2u, \ldots$

In our case, with $u_1 = (1,1,1,1,1)$ we have

$u_2 = \frac{Bu_1}{15} \approx (1,2,3,3,3)$ and

$B^2u_2 \approx (41, 85, 116, 134, 146)$

So we expect ~~the big~~ $\|B\| \approx 44$

and the eigenvector to be approximately equal to $(41, 85, 116, 134, 146)$.

In fact, Wolfram alpha, with

eigenvalues {{1,2,3,4,5},{2,6,7,8,9},{3,7,10,11,12},{4,8,11,13,14},{5,9,12,14,15}}

produces eigenvalues

$\lambda_1 \approx 44.2126$, $\lambda_3 \approx 1.26641$, $\lambda_5 \approx 0.241956$

$\lambda_2 \approx -1.43905$, $\lambda_4 \approx 0.718115$, and the eigenvector corresponding to the largest eigenvalue is

$$v_1 \approx (0.282093, 0.585915, 0.788872, 0.91288, 1)$$

Our computation (by hand)

$$v_1 \approx \frac{Bu_4}{146} = \left(\frac{41}{146}, \frac{85}{146}, \frac{116}{146}, \frac{134}{146}, 1\right)$$

$$\approx (0.2808, 0.58219, 0.7945, 0.9178, 1)$$

where the last line was done by calculator.

So, by hand we were able to get a very good approximation (after only two steps!).

We had estimated

$\|W\| \approx 44$ and Wolfram alpha estimates

$\|W\| \approx 44.2126.$

<u>For computing the norm $\|T\|$</u>

Let

$$C = \bar{A}^t A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 14 & 15 \\ 16 & 0 & 2 & 0 & 20 \\ 1 & 0 & 3 & 4 & 10 \end{pmatrix}^t \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 14 & 15 \\ 16 & 0 & 2 & 0 & 20 \\ 1 & 0 & 3 & 4 & 10 \end{pmatrix}$$

Then C is selfadjoint and since, if $\bar{A}^t A u = \gamma u$ use

$$\|Au\|^2 = \langle Au, Au\rangle = \langle u, \bar{A}^t A u\rangle = \langle u, \gamma u\rangle = \bar{\gamma}\|u\|^2 = \gamma\|u\|^2$$

~~then~~ to show

$\|A\| = \sqrt{\gamma}$, where γ is the largest eigenvalue of C

As we did for computing the norm $\|W\|$ we know

$$\frac{\langle Cu, u\rangle}{\|u\|^2} \quad \text{with } u = (1,1,1,1,1) \text{ and } \|u\| = \sqrt{5}$$

is likely not a bad estimate of $\|C\|$.

So

$$\|A\| \approx \sqrt{\frac{\langle Cu, u\rangle}{\|u\|^2}} = \frac{\sqrt{\langle Au, Au\rangle}}{\sqrt{5}}$$

and

$$Au = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 14 & 15 \\ 16 & 0 & 2 & 0 & 20 \\ 1 & 0 & 3 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 40 \\ 52 \\ 38 \\ 18 \end{pmatrix}$$

and

$$\langle Au, Au\rangle = 15^2 + 40^2 + 52^2 + 38^2 + 18^2$$

so

$$\frac{\sqrt{\langle Au, Au\rangle}}{\|u\|} = \cancel{\frac{1}{\sqrt{5}}\sqrt{15^2+40^2}} \sqrt{\frac{15^2+40^2+52^2+38^2+18^2}{5}}$$

$$\approx \sqrt{5\cdot 3 + 8\cdot 40 + 10\cdot 52 + 7\cdot 38 + 3\cdot 18}$$

$$\approx \sqrt{15 + 320 + 520 + 280 + 60}$$

$$\approx \sqrt{1200} \approx 10\cdot\sqrt{12} \approx 10\cdot 3.7 \approx 37.$$

In fact Wolfram alpha says

$$C = \begin{pmatrix} 415 & 176 & 86 & 216 & 560 \\ 176 & 147 & 62 & 239 & 260 \\ 86 & 62 & 86 & 96 & 165 \\ 216 & 239 & 96 & 309 & 360 \\ 560 & 260 & 165 & 360 & 450 \end{pmatrix}$$

with largest eigenvalue 1537.87 and

$$\sqrt{1537.87} \approx 39.216.$$

So our estimate of 37 was not too bad