Metric and Hilbert Spaces: Lecture 3

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Let $T: V \to W$ be a linear transformation.

The <u>adjoint</u> is the linear transformation

$$T^*: W^* \to V^* \text{ given by } (T^*\varphi)(v) = \varphi(Tv)$$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & & \downarrow \varphi \\ & & \mathbb{K} \end{array}$$

Assume that $(V, \langle\,,\rangle_V)$ and $(W, \langle\,,\rangle_W)$ are Hilbert spaces so that the Riesz representation theorem gives bijections

$$\begin{array}{rcl} \Phi: V & \to & V^* \\ v & \mapsto & \varphi_v \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi: W & \to & W^* \\ w & \mapsto & \psi_w \end{array}$$

where

$$\begin{array}{rcl} \varphi_v: V & \to & \mathbb{K} \\ x & \mapsto & \langle x, v\rangle \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi_w: W & \to & \mathbb{K} \\ y & \mapsto & \langle y, w\rangle. \end{array}$$

Define $T^*: W \to V$ by $T^*w = (\Phi^{-1} \circ T^* \circ \Psi)(w)$

$$\begin{array}{ccc} W & \xrightarrow{T^*} & V \\ \Psi \downarrow & & \uparrow \Phi^{-1} \\ W^* & \xrightarrow[T^*]{} & V^* \end{array}$$

So $\Phi(\Psi^*_w) = T^*(\Phi(w))$ for $w \in W$.

So $\varphi_{\Psi^*_w} = T^*_{\varphi_w}$, for $w \in W$.

So $\varphi_{\Psi^*_w}(y) = (T^*\varphi_w)(y)$, for $w \in W$ and $y \in V$.

So $\langle y, \Psi^*_w \rangle = \varphi_{\Psi^*_w}(y) = (T^*\varphi_w)(y) = \varphi_w(Ty)$

$= \langle Ty, w \rangle$, for $w \in W$ and $y \in V$.

Let $T^*: W^* \to V^*$ be the adjoint of T. 10.10.2017 A. Ram UniMelb. (3)

Is $\|T^*\| = \|T\|$?

If $\varphi \in W^*$ then
$$\|T^*\varphi\| = \sup\left\{ \frac{\|(T^*\varphi)(v)\|}{\|v\|} \;\Big|\; v \in V \right\}.$$

So
$$|T^*\varphi(v)| = |\varphi(Tv)| \le \|\varphi\| \cdot \|Tv\| \le \|\varphi\| \cdot \|T\| \cdot \|v\|.$$

So $\quad \|T^*\varphi\| \le \|\varphi\| \cdot \|T\|.$

So $\quad \|T^*\| \le \|T\|.$

If $(V, \langle\,,\rangle)$ is a Hilbert space then $T = (T^*)^*$.

So $\quad \|(T^*)^*\| \le \|T^*\|$

So $\quad \|T\| \le \|T^*\|.$