27.07.2017
A. Ram (1)

Metric and Hilbert Spaces: Lecture 2

cover compact	ball compact	bounded
sequentially compact	Cauchy compact	closed
topological spaces	uniform spaces	metric spaces

A topological space is a set X with a collection $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U \in \mathcal{S}} U\right) \in \mathcal{T}$

(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then
$$U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}$$

Let $(X, \mathcal{T})$ be a topological space.
An open set in X is a subset $A \subseteq X$ such that $A \in \mathcal{T}$.

Every metric space can be made into a topological space

Let (X,d) be a metric space.

Let $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$.

The open ball of radius ε at x is

$$B_\varepsilon(x) = \{ y \in X \mid d(y,x) < \varepsilon \}.$$

Let

$$\mathcal{B} = \{ B_\varepsilon(x) \mid \varepsilon \in \mathbb{R}_{>0} \text{ and } x \in X \},$$

$$\mathcal{T} = \left\{ U \subseteq X \;\middle|\; \begin{array}{c} \text{there exists } \mathcal{S} \subseteq \mathcal{B} \text{ such that} \\ U = \left(\bigcup_{S \in \mathcal{S}} S \right) \end{array} \right\}.$$

Theorem $(X, \mathcal{T})$ is a topological space.

English: FOR a METRIC SPACE (X,d).

$\mathcal{B}$ is the set of open balls on X

$\mathcal{T}$ is the metric space topology on X

A set U is open in X if

U is a union of open balls.

Uniform spaces

A uniform space is a set X with a collection $\mathcal{X}$ of subsets of $X \times X$ such that

(a) If $E \in \mathcal{X}$ then $E \supseteq \Delta(X)$

(b) If $E \in \mathcal{X}$ and $D \subseteq X \times X$ and $D \supseteq E$ then $D \in \mathcal{X}$

(c) If $\ell \in \mathbb{Z}_{>0}$ and $E_1, E_2, \ldots, E_\ell \in \mathcal{X}$ then
$$E_1 \cap E_2 \cap \cdots \cap E_\ell \in \mathcal{X}$$

(d) If $E \in \mathcal{X}$ then $\sigma(E) \in \mathcal{X}$
$$\sigma(E) = \{(y,x) \mid (x,y) \in E\}.$$

(e) If $E \in \mathcal{X}$ then there exists $D \in \mathcal{X}$ with
$$D \times_X D \subseteq E,$$

$$D \times_X D = \left\{ (x,y) \;\middle|\; \begin{array}{l} \text{there exists } z \in X \text{ with} \\ (x,z) \in D \text{ and } (z,y) \in D \end{array} \right\}$$

Every metric space can be made into a uniform space

Let (X,d) be a metric space.

Let $\varepsilon \in \mathbb{R}_{>0}$. The ε-diagonal in X is

$$B_\varepsilon = \{(x,y) \in X \times X \mid d(x,y) < \varepsilon\}$$

Let

$$\mathcal{X} = \left\{ E \subseteq X \times X \;\middle|\; \begin{array}{c}\text{there exists } \varepsilon \in \mathbb{R}_{>0} \text{ with} \\ B_\varepsilon \subseteq E\end{array} \right\}$$

Theorem $(X, \mathcal{X})$ is a uniform space.