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Metric and Hilbert Spaces: Lecture 29
Examples of linear operators

A <u>bounded linear operator</u> <u>from V to W</u> is a linear transformation $T: V \to W$ such that
there exists $C \in \mathbb{R}_{>0}$ such that
if $u \in V$ then $\|Tu\| \le C\|u\|$.

The <u>norm of T</u> is the minimal C that works.

<u>Examples</u>

(1) Let $(V, \|\ \|)$ be a normed vector space and

$$\begin{array}{rcl} 1: V & \to & V \\ x & \mapsto & x \end{array} \qquad \text{and} \qquad \begin{array}{rcl} 0: V & \to & V \\ x & \mapsto & 0 \end{array}$$

Then $\|1\| = 1$ and $\|0\| = 0$.

(2) Let $C[a,b] = \{ f : \mathbb{R}_{[a,b]} \to \mathbb{R} \mid f \text{ is continuous} \}$
with the sup norm

$$\|f\|_\infty = \sup\{ |f(t)| \mid t \in [a,b] \}.$$

Let $T: C[a,b] \to \mathbb{R}$ be given by

$$Tf = \int_a^b f(t)\,dt$$

If $g:\mathbb{R}_{[a,b]} \to \mathbb{R}$ is the function given by $g(t)=1$ then

$$Tg = \int_a^b dt = b-a \quad \text{and} \quad \|Tg\| = b-a = (b-a)\|g\|_\infty$$

since $\|g\|_\infty = 1$. So $\|T\| \geq b-a$.

If $f \in C[a,b]$ then

$$|Tf| = \left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt \leq \|f\|_\infty (b-a)$$

So $\|T\| \leq b-a$.

Thus $\|T\| = b-a$.

(3) <u>Integral operators</u> Let

$$C[a,b] = \{f:\mathbb{R}_{[a,b]} \to \mathbb{C} \mid f \text{ is continuous}\}$$

with norm given by

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in [a,b]\}$$

Let $K:\mathbb{R}_{[a,b]} \times \mathbb{R}_{[a,b]} \to \mathbb{C}$ be a continuous function.

Define $T: \mathbb{C}[a,b] \to C[a,b]$ by

$$(Tf)(t) = \int_a^b K(t,s) f_s \, ds$$

(generalized matrix multiplication!).

To show: (a) If $f \in C[a,b]$ then $Tf \in C[a,b]$.

(b) T is a bounded linear operator.

(a) Assume $f \in C[a,b]$.

To show: Tf is continuous.

In fact we will show: Tf is uniformly continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $t, t' \in \mathbb{R}_{[a,b]}$ and $|t-t'|<\delta$ then

$$|Tf(t) - Tf(t')| < \varepsilon.$$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Since $\mathbb{R}_{[a,b]} \times \mathbb{R}_{[a,b]}$ is compact and K is continuous then K is uniformly continuous.

Let $\delta \in \mathbb{R}_{>0}$ be such that

if $s, s', t, t' \in \mathbb{R}_{[a,b]}$ and $d((s,t),(s',t'))<\delta$

then $|K(s,t) - K(s',t')| < \dfrac{\varepsilon}{(b-a)\|f\|_\infty}$.

To show: If $t, t' \in \mathbb{R}_{[a,b]}$ and $|t-t'|<\delta$ then

$$|Tf(t) - Tf(t')| < \varepsilon.$$

Assume $t, t' \in \mathbb{R}_{[a,b]}$ and $|t-t'| < \delta$.

To show: $|Tf(t) - Tf(t')| < \varepsilon$.

$$|Tf(t) - Tf(t')| = \left| \int_a^b (K(s,t) - K(s,t'))f(s)\,ds \right|$$

$$\leq \int_a^b |K(s,t) - K(s,t')| \cdot |f(s)|\,ds$$

$$< \frac{\varepsilon}{(b-a)\|f\|_\infty} \cdot (b-a)\|f\|_\infty = \varepsilon$$

So Tf is uniformly continuous.

So Tf is continuous and $Tf \in C[a,b]$.

(b) To show: T is a bounded linear operator.

To show: There exists $C \in \mathbb{R}_{>0}$ such that

if $f \in C[a,b]$ then $\|Tf\|_\infty \leq C\|f\|_\infty$.

Since $\mathbb{R}_{[a,b]} \times \mathbb{R}_{[a,b]}$ is compact and K is continuous then

$K(\mathbb{R}_{[a,b]} \times \mathbb{R}_{[a,b]})$ is compact and

$K(\mathbb{R}_{[a,b]} \times \mathbb{R}_{[a,b]})$ is bounded.

Let

$$C = \sup\{ |K(s,t)| \mid s,t \in \mathbb{R}_{[a,b]} \}.$$

To show: If $f \in C[a,b]$ then $\|Tf\|_\infty \le C\|f\|_\infty$.

Assume $f \in C[a,b]$.

Then
$$|Tf(t)| \le \int_a^b |K(s,t)| \cdot |f(s)|\, ds \le (b-a)C\|f\|_\infty.$$

So $\|Tf\|_\infty = \sup\{|T(f(t))| \; t \in \mathbb{R}_{[a,b]}\} \le (b-a)C\|f\|_\infty$.

So $\|T\| \le (b-a)\cdot C$

So $\|T\|$ is bounded.