Metric and Hilbert Spaces Lecture 26

21.09.2017
A. Ram
Uni Melb
①

Let $(H, \langle\,\rangle)$ be an inner product space and

$$T: H \to H \text{ a bounded linear operator}$$

The λ-eigenspace of T is

$$X_\lambda = \{ v \in H \mid Tv = \lambda v \}$$

The point spectrum is $\sigma_p(T) = \{ \lambda \in \mathbb{C} \mid X_\lambda \neq 0 \}$

How do we find eigenvectors? Let

$$\lambda = \sup \{ |\langle Tu, u\rangle| \mid u \in H \text{ and } \|u\| = 1 \}.$$

Claim: $\lambda = \|T\|$.

Sketch of proof: To show: (a) $\|T\| \leq \lambda$

(b) $\|T\| \geq \lambda$.

(b) Use Cauchy-Schwarz

(a) Expand $|\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle|$

where $x \in H$ with $\|x\| = 1$ and

$$y = \frac{Tx}{\|Tx\|}.$$

Next: Let $(u_1, u_2, \ldots)$ be a sequence in $\{ u \in H \mid \|u\| = 1 \}$ such that

$$\lim_{n \to \infty} |\langle Tu_n, u_n\rangle| = \lambda.$$

Claim: $\lim_{n \to \infty} (T - \lambda) u_n = 0$

Sketch of proof: Expand $\|Tu_n - \lambda u_n\|^2$.

<u>Next</u>: <u>Assume T is compact</u>

Let y be a cluster point of $(Tu_1, Tu_2, \ldots)$.

Let $(u_{n_1}, u_{n_2}, \ldots)$ be a subsequence such that $(Tu_{n_1}, Tu_{n_2}, \ldots)$ converges to y.

<u>Claim</u>: (a) $\lim\limits_{k\to\infty} \lambda u_{n_k} = y$

Sketch of proof: Use that

$$\lim_{n\to\infty} (T-\lambda)u_n = 0 \text{ and } \lim_{k\to\infty} Tu_{n_k} = y.$$

(b) $\|y\| = \|T\|$

(c) $Ty = \|T\| y$

(d) $\dfrac{\langle Ty, y\rangle}{\|y\|^2} = \|T\|$.

Sketch of proof: Use

$\lim\limits_{k\to\infty} \lambda u_{n_k} = y$ to get $\|y\| = |\lambda|$

Use $\lim\limits_{k\to\infty} \lambda u_{n_k} = y$ to show $\|Ty - \lambda y\| = 0$

Finally, just compute $\langle Ty, y\rangle$.

The Rayleigh quotient.

Making the process less existential. Let

$b_0 \in H$. with $\|b_0\| = 1$.

$$b_1 = \frac{Ab_0}{\|Ab_0\|} \qquad \mu_1 = \frac{\langle Ab_0, b_0\rangle}{\|b_0\|^2}$$

$$b_2 = \frac{Ab_1}{\|Ab_1\|} \qquad \mu_2 = \frac{\langle Ab_1, b_1\rangle}{\|b_1\|^2}$$

$$\vdots \qquad\qquad \vdots$$

Then $b_k = \dfrac{A^k b_0}{\|A^k b_0\| \cdot \|A^{k-1} b_0\| \cdots \|Ab_0\|}$

Do

$\lim_{k\to\infty} b_k = b$ and $\lambda = \lim_{k\to\infty} \mu_k$ exist?

Sometimes yes and sometimes no.

When yes and when no?

When they do, b is an eigenvector of eigenvalue λ.

The fundamental theorem of algebra

Find a root of

$$x^6 - 3x^5 + 7x^4 + 2x^3 - 3x^2 + 5x + 4.$$

Find an eigenvalue of

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$