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Metric and Hilbert Spaces: Lecture 25

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Let H be a $\mathbb{C}$-vector space and let

$T: H \to H$ be a linear transformation.

Let $\lambda \in \mathbb{C}$. The <u>λ-eigenspace</u> of T is

$$X_\lambda = \{v \in H \mid Tv = \lambda v\} = \ker(T-\lambda).$$

The <u>point spectrum</u> of T is

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid X_\lambda \neq 0\}$$

Let $(V, \|\ \|)$ be a normed vector space and let

$T: V \to V$ be a bounded linear operator.

The operator $T: V \to V$ is <u>compact</u> if T satisfies:

if $(x_1, x_2, \ldots)$ is a sequence in $\{x \in H \mid \|x\| = 1\}$

then $(Tx_1, Tx_2, \ldots)$ has a cluster point in V.

Let $(H, \langle\ \rangle)$ be an inner product space and let

$T: H \to H$ be a bounded linear operator.

T is <u>self adjoint</u> if T satisfies:

if $x, y \in H$ then $\langle Tx, y\rangle = \langle x, Ty\rangle$.

T is an <u>isometry</u> if T satisfies

if $x, y \in H$ then $\langle Tx, Ty\rangle = \langle x, y\rangle$

T is <u>unitary</u> if T is an isometry and T is invertible.

T is <u>positive</u> if T is self adjoint and if $x \in H$ then $\langle Tx, x \rangle \in \mathbb{R}_{\geq 0}$.

<u>Proposition</u> (a) Let H be a $\mathbb{C}$-vector space and $T: H \to H$ a linear transformation.
Then
T has an eigenvector of eigenvalue λ if and only if $\lambda - T$ is not injective.

(b) Let $T: H \to H$ be a compact linear operator.
Then
$\lambda - T$ is injective if and only if $\lambda - T$ is bijective.

<u>Proof</u>

(a) $X_\lambda \neq 0$ if and only if $\ker(\lambda - T) \neq 0$
if and only if $\lambda - T$ is not injective.

Recall
$$\ker(\lambda - T) = \{v \in H \mid (\lambda - T)v = 0\}.$$

(b) $\lambda - T$ is bijective implies $\lambda - T$ is injective.
To show: If T is compact and $\lambda - T$ is injective then $\lambda - T$ is surjective.

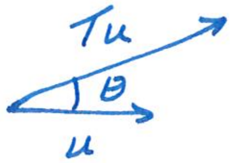
See Bressan Theorem 6.1. $\square$

If $T: H \to H$ is a self-adjoint operator and $u \in H$ then

$$\langle Tu, u\rangle = \langle u, Tu\rangle = \overline{\langle Tu, u\rangle} \text{ so that } \langle Tu, u\rangle \in \mathbb{R}.$$

The Cauchy-Schwarz inequality gives

$$|\langle Tu, u\rangle| \leq \|Tu\| \cdot \|u\| \text{ and } \theta = \cos^{-1}\left(\frac{\langle Tu, u\rangle}{\|Tu\| \cdot \|u\|}\right)$$



If $\theta = 0$ or π $\left(\text{i.e. } \cos\theta = \frac{\langle Tu, u\rangle}{\|Tu\| \cdot \|u\|} = \pm 1\right)$

then $|\langle Tu, u\rangle|$ is achieving its maximum, and u is an eigenvector!!

Theorem

Let $(H, \langle,\rangle)$ be a Hilbert space and let $T: H \to H$ a bounded self adjoint linear operator.

Then
$$\|T\| = \sup\left\{ |\langle Tu, u\rangle| \;\middle|\; \begin{matrix} u \in H \text{ and} \\ \|u\| = 1 \end{matrix} \right\}$$

Proof Let $\lambda = \sup\{ |\langle Tu, u\rangle| \mid u \in H \text{ and } \|u\| = 1 \}$.

To show: $\|T\| = \lambda$.

To show: (a) $\|T\| \geq \lambda$

(b) $\|T\| \leq \lambda$.

(a) Assume $u \in H$ and $\|u\| = 1$.

By Cauchy-Schwarz,
$$|\langle Tu, u\rangle| \leq \|Tu\| \cdot \|u\| \leq \|T\| \cdot \|u\| \cdot \|u\| = \|T\|.$$

So $\|T\| \geq \lambda$.

(b) Let $x \in H$ with $\|x\| = 1$. Let
$$y = \frac{Tx}{\|Tx\|} \text{ so that } \|y\| = 1.$$

Since T is self adjoint then $\langle Tx, y\rangle \in \mathbb{R}$ and

$$\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle$$
$$= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle$$
$$- \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle - \langle Ty, y\rangle$$
$$= 4\langle Tx, y\rangle = 4\frac{\langle Tx, Tx\rangle}{\|Tx\|} = 4\|Tx\|$$

Then

$$4\|Tx\| = |\langle T(x+y), x+y\rangle - \langle T(x-y), (x-y)\rangle|$$

$$\leq |\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|$$

$$= \left|\left\langle \frac{T(x+y)}{\|x+y\|}, \frac{x+y}{\|x+y\|}\right\rangle\right| \|x+y\|^2 + \left|\left\langle \frac{T(x-y)}{\|x-y\|}, \frac{x-y}{\|x-y\|}\right\rangle\right| \|x-y\|^2$$

$$\leq \lambda \|x+y\|^2 + \lambda \|x-y\|^2$$

$$= \lambda(2\|x\|^2 + 2\|y\|^2) = 4\lambda.$$

So $\|T\| \leq \lambda$. //