Metric and Hilbert Spaces: Lecture 24

15.09.2017
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Orthonormal sequences

Let $(H, \langle\, \rangle)$ be a Hilbert space.

An <u>orthonormal</u> <u>sequence</u> in H is a sequence $(a_1, a_2, \ldots)$ in H such that

if $i, j \in \mathbb{Z}_{>0}$ then $\langle a_i, a_j \rangle = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$

<u>Theorem</u> Let $(H, \langle\, \rangle)$ be a Hilbert space. Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H. Let

$W = \mathbb{K}\text{-span}\{a_1, a_2, \ldots\}$, $\overline{W}$ the closure of W in H

and $P_{\overline{W}}: H \to H$ the projection onto $\overline{W}$.

If $x \in H$ then

$$P_{\overline{W}}(x) = \sum_{n \in \mathbb{Z}_{>0}} \langle x, a_n \rangle a_n.$$

<u>Proof</u> Step 1. (Bessel's inequality) If $x \in H$ then

$$\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^2 \leq \|x\|^2$$

Step 2: If $x \in H$ then $P(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle a_n$ exists in H.

Step 3: If $x \in H$ then $P(x) \in \overline{W}$

Step 4: If $x \in H$ then $x - P(x) \in (\overline{W})^\perp$.

Assume $r, s \in \mathbb{Z}_{\geq N}$.

To show: $\|x_r - x_s\|^2 < \varepsilon^2$

$$\|x_r - x_s\|^2 = \Big\| \sum_{j=1}^{r} \langle x, a_j\rangle a_j - \sum_{j=1}^{s} \langle x, a_j\rangle a_j \Big\|^2$$

$$= \Big\| \sum_{j=r+1}^{s} \langle x, a_j\rangle a_j \Big\|^2 = \sum_{j=r+1}^{s} |\langle x, a_j\rangle|^2$$

$$= \big| \|x_s\|^2 - \|x_r\|^2 \big| = \big| \|x_s\|^2 - y^2 + y^2 - \|x_r\|^2 \big|$$

$$\leq \big| \|x_s\|^2 - y^2 \big| + \big| y^2 - \|x_r\|^2 \big| \leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.$$

So $(x_1, x_2, \ldots)$ is a Cauchy sequence in H.

So $\lim_{k\to\infty} x_k$ exists in H.

So $\sum_{j=1}^{k} \langle x, a_j\rangle a_j$ exists in H.

<u>Step 3</u>: To show: $\sum_{n=1}^{\infty} \langle x, a_n\rangle a_n \in \overline{W}$

Since $x_k = \sum_{j=1}^{k} \langle x, a_j\rangle a_j$ is an element of

$\mathbb{K}\text{-span}\{a_1, a_2, \ldots\} = W$ then

$$P(x) = \lim_{k\to\infty} x_k \in \overline{W}$$

Step 2: To show: If $x \in H$ then
$$P(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle a_n \text{ exists in } H.$$

Assume $x \in H$. Let $x_k = \sum_{n=1}^{k} \langle x, a_n \rangle a_n$

To show: $\lim_{k\to\infty} x_k$ exists in H.

Using that H is complete,
To show: $(x_1, x_2, \ldots)$ is a Cauchy sequence in H.

We know that $\|x_k\| = \sum_{n=1}^{k} |\langle x, a_n \rangle|^2$ so that,
by Bessel's inequality,
$(\|x_1\|, \|x_2\|, \ldots)$ is an increasing sequence in $\mathbb{R}_{\geq 0}$
bounded by $\|x\|$.

So $(\|x_1\|, \|x_2\|, \ldots)$ converges in $\mathbb{R}_{\geq 0}$.

Let $y = \lim_{k\to\infty} \|x_k\|$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$
such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $N \in \mathbb{Z}_{>0}$ such that
if $r, s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\|^2 < \varepsilon^2$

Let $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N}$ then $|y^2 - \|x_k\|^2| < \frac{\varepsilon^2}{2}$.

Step 5: If $x \in H$ then $P(x) = P_{\overline{W}}(x)$.

Step 1: To show: $\lim_{k\to\infty}\left(\sum_{n=1}^{k}|\langle x, a_n\rangle|^2\right) \le \|x\|^2$.

Assume $k \in \mathbb{Z}_{\ge 0}$. Let

$$x_k = \sum_{n=1}^{k}\langle x, a_n\rangle a_n \text{ so that}$$

$$\|x_k\|^2 = \sum_{n=1}^{k}\langle x, a_n\rangle\overline{\langle x, a_n\rangle} = \sum_{n=1}^{k}|\langle x, a_n\rangle|^2.$$

To show: $\|x_k\|^2 \le \|x\|^2$.

Then

$$\langle x - x_k, x_k\rangle = \langle x, x_k\rangle - \langle x_k, x_k\rangle$$
$$= \sum_{n=1}^{k}\langle x, a_n\rangle\overline{\langle x, a_n\rangle} - \sum_{n=1}^{k}\langle x, a_n\rangle\overline{\langle x, a_n\rangle} = 0,$$

and

$$\|x\|^2 = \langle x, x\rangle = \langle x_k + (x - x_k), x_k + (x - x_k)\rangle$$
$$= \langle x_k, x_k\rangle + \langle x_k, x - x_k\rangle + \langle x - x_k, x_k\rangle + \langle x - x_k, x - x_k\rangle$$
$$= \|x_k\|^2 + 0 + 0 + \|x - x_k\|^2$$

So $\|x_k\|^2 \le \|x\|^2$.

So $\sum_{n=1}^{\infty}|\langle x, a_n\rangle|^2 = \lim_{k\to\infty}\left(\sum_{n=1}^{k}|\langle x, a_n\rangle|^2\right) = \lim_{k\to\infty}\|x_k\|^2 \le \|x\|^2$.

Step 4: To show: $x - P(x) \in W^\perp$.

To show: If $b \in \overline{W}$ then $\langle x - P(x), b\rangle = 0$.

Assume $b \in \overline{W}$.

Let $(b_1, b_2, \ldots)$ be a sequence in W with $\lim_{n\to\infty} b_n = b$.

To show: $\langle x - P(x), b\rangle = 0$.

Using that $\langle x - P(x), \cdot\rangle : H \to \mathbb{K}$ is continuous

$$\langle x - P(x), b\rangle = \langle x - P(x), \lim_{n\to\infty} b_n\rangle = \lim_{n\to\infty}\langle x - P(x), b_n\rangle.$$

Since $b_n \in W$ there exists $\ell \in \mathbb{Z}_{>0}$ and $c_1, c_2, \ldots, c_\ell \in \mathbb{K}$ such that

$$b_n = c_1 a_1 + \cdots + c_\ell a_\ell \quad \text{and}$$

$$\langle x - P(x), b_n\rangle = \sum_{r=1}^{\ell} \bar{c}_r \langle x - P(x), a_r\rangle.$$

Using that $\langle \cdot, a_r\rangle : H \to \mathbb{K}$ is continuous and

$\langle x_k, a_r\rangle = \langle x, a_r\rangle$ for $k \in \mathbb{Z}_{\geq r}$ then

$$\langle x - P(x), a_r\rangle = \langle x, a_r\rangle - \langle P(x), a_r\rangle$$

$$= \langle x, a_r\rangle - \langle \lim_{k\to\infty} x_k, a_r\rangle$$

$$= \langle x, a_r\rangle - \lim_{k\to\infty}\langle x_k, a_r\rangle = \langle x, a_r\rangle - \langle x, a_r\rangle = 0.$$

So $\langle x - P(x), b_n\rangle = \sum_{j=1}^{\ell} \bar{c}_j \langle x - P(x), a_j\rangle = 0$.

So $\langle x - P(x), b\rangle = \lim_{n\to\infty}\langle x - P(x), b_n\rangle = \lim_{n\to\infty} 0 = 0$

So $x-P(x) \in (\overline{W})^\perp$

Step 5: Since $x-P(x) \in \overline{W}^\perp$ ~~then~~ and $x-P_{\overline{W}}(x) \in \overline{W}^\perp$

and $P(x)-P_{\overline{W}}(x) \in \overline{W}$ then

$\cancel{\|P(x)+P_{\overline{W}}(x)\|^2 = \|P(x)-x}$

$$P(x)-P_{\overline{W}}(x) = (x-P_{\overline{W}}(x)) - (x-P(x)) \in \overline{W}^\perp \cap \overline{W} = \{0\}.$$

So $P(x) = P_{\overline{W}}(x)$.