<u>Metric and Hilbert spaces</u>: Lecture 22 12/09/2017 UniMelb A. Ram (1)

<u>Projections</u>

<u>Theorem</u> Let $(H, \langle\,\rangle)$ be a Hilbert space and let $W$ be a closed subspace of $H$.

Let
$$d(x,W) = \inf\{d(x,w) \mid w \in W\}$$

(a) If $x \in H$ then there exists a unique $y \in W$ such that $d(x,y) = d(x,W)$.

(b) Define $P_W : H \to H$ by $P_W(x) = y$, where $y \in W$ such that $d(x,y) = d(x,W)$.

Then
$$P_W(x) \in W, \quad (id - P_W)(x) \in W^\perp, \quad \|P_W\| = 1,$$
$$P_W^2 = P_W, \quad (id - P_W)^2 = id - P_W, \quad id = P_W + (id - P_W).$$

<u>Proof</u> (a) Assume $x \in H$.

To show: (aa) There exists $y \in W$ such that $d(x,y) = d(x,W)$

(ab) If $y, y' \in W$ and $d(x,y) = d(x,y') = d(x,W)$ then $y = y'$

(aa) Let $(y_1, y_2, \ldots)$ be a sequence in $W$ such that

$$d(x, y_n)^2 \le D^2 + \frac{1}{10^n} \quad \text{where } D = d(x, W).$$

To show: There exists $y \in W$ such that $\lim_{n\to\infty} y_n = y$.

Using that $W$ is closed,

To show: There exists $y \in H$ such that $\lim_{n\to\infty} y_n = y$.

Using that $H$ is complete,

To show: $(y_1, y_2, \ldots)$ is Cauchy.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ then $\|y_m - y_n\| < \varepsilon$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ then $\|y_m - y_n\| < \varepsilon$.

Let $N \in \mathbb{Z}_{>0}$ such that $\frac{1}{10^{N-1}} < \varepsilon^2$.

To show: If $m, n \in \mathbb{Z}_{\ge N}$ then $\|y_m - y_n\| < \varepsilon$

Assume $m, n \in \mathbb{Z}_{\ge N}$.

To show: $\|y_m - y_n\| < \varepsilon$

To show: $\|y_m - y_n\|^2 < \varepsilon^2$.

Since, by the parallelogram identity,

$$4D^2 + \|y_m - y_n\|^2 \le 4\|x - \tfrac{1}{2}(y_n + y_m)\|^2 + \|y_n - y_m\|^2$$

$$= \|(x-y_m)+(x-y_n)\|^2 + \|(x-y_m)-(x-y_n)\|^2$$

$$= 2\|x-y_m\|^2 + 2\|x-y_n\|^2$$

$$\le 2\left(D^2 + \frac{1}{10^m}\right) + 2\left(D^2 + \frac{1}{10^n}\right)$$

$$\le 2D^2 + \frac{2}{10^N} + 2D^2 + \frac{2}{10^N} \le 4D^2 + \frac{40}{10^N}$$

$$= 4D^2 + \frac{1}{10^{N-1}} < 4D^2 + \varepsilon^2$$

then $\|y_m - y_n\|^2 < \varepsilon^2$.

So $\|y_m - y_n\| < \varepsilon$

So $(y_1, y_2, \dots)$ is a Cauchy sequence.

So $(y_1, y_2, \dots)$ converges and the limit $y = \lim_{n\to\infty} y_n$ is in $W$.

(ab) Assume $y, y' \in W$ and $d(x,y) = d(x,y') = d(x,W)$.

Since $\frac{1}{2}(y+y') \in W$ gives that

$$d(x, \tfrac{1}{2}(y+y')) \ge d(x,W)$$

the parallelogram identity gives

$$\|y-y'\|^2 = 2\|x-y\|^2 + 2\|x-y'\|^2 - 4\|x - \tfrac{1}{2}(y+y')\|^2$$

$$= 2\,d(x,y)^2 + 2\,d(x,y')^2 - 4\,d(x, \tfrac{1}{2}(y+y'))^2$$

$$\leq 2d(x,W)^2 + 2d(x,W)^2 - 4\,d(x,W)^2 = 0$$

So $y = y'$!

(b) <u>Part 1</u>: To show: $x - P_W(x) \in W^\perp$.

Let $y = P_W(x)$. Let $w \in W$.

By (a), the function $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(\lambda) = \|x - (y + \lambda w)\|^2 = \|x-y\|^2 + |\lambda|^2\|w\|^2 + 2Re(\langle x-y, \lambda w\rangle)$$

has a unique global minimum at $\lambda = 0$.

So $f'(0) = 0$.

So $Re(\langle x-y, \lambda w\rangle) = 0$.

Similarly $Re(\langle x-y, iw\rangle) = 0$.

So $\langle x-y, w\rangle = 0 + 0i$.

So $x - y \in W^\perp$.

<u>Part 2</u>: To show: $P_W$ and $id - P_W$ are linear operators.

Assume $v_1, v_2 \in H$ and $c_1, c_2 \in \mathbb{K}$.

Then

$c_1 P_W(v_1) + c_2 P_W(v_2) \in W$ and

$x = c_1 v_1 + c_2 v_2 - (c_1 P_W(v_1) + c_2 P_W(v_2)) \in W^\perp$.

So $P_W(x) = 0$.

So $P_W(c_1 v_1 + c_2 v_2) = c_1 P_W(v_1) + c_2 P_W(v_2)$.

So $P_W$ is linear.

So $P_W$ and $id - P_W$ are linear operators.

<u>Part 3</u>: To show: $\|P_W\| = 1$.

Assume $v \in H$ and $v \neq 0$.

Since $\langle P_W(v), (id - P_W)(v)\rangle = 0$ then the Pythagorean theorem gives

$$\frac{\|P_W(v)\|^2}{\|v\|^2} \le \frac{\|P_W(v)\|^2 + \|(id-P_W)(v)\|^2}{\|v\|^2} = \frac{\|v\|^2}{\|v\|^2} = 1.$$

So $\|P_W\| \le 1$.

If $w \in W$ then $P_W(w) = w$ so that

$$\|P_W(w)\| = \|w\| \quad \text{and} \quad \|P_W\| \ge 1.$$

So $\|P_W\| = 1$.

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