Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let V be a $\mathbb{K}$-vector space.

A <u>basis</u> of V is a subset $B\subseteq V$ such that

(a) $\mathbb{K}\text{-span}(B)=V$

(b) B is linearly independent.

where

$$\mathbb{K}\text{-span}(B)=\left\{a_1b_1+\cdots+a_\ell b_\ell \;\middle|\; \begin{array}{l}\ell\in\mathbb{Z}_{>0},\ b_1,\ldots,b_\ell\in B\\ a_1,\ldots,a_\ell\in\mathbb{K}\end{array}\right\}$$

and B is <u>linearly independent</u> means B satisfies

if $\ell\in\mathbb{Z}_{>0}$ and $b_1,\ldots,b_\ell\in B$ and $a_1,\ldots,a_\ell\in\mathbb{K}$ and $a_1b_1+\cdots+a_\ell b_\ell=0$ then $a_1=0, a_2=0,\ldots,a_\ell=0$.

Let V be a topological vector space.

A <u>topological basis of V</u> is a set $B\subseteq V$ such that

(a) $\overline{\mathbb{K}\text{-span}(B)}=V$

(b) B is linearly independent.

A <u>Schauder basis of V</u> is a sequence $(b_1,b_2,\ldots)$ in V such that

if $v\in V$ then there exists a unique sequence $(a_1,a_2,\ldots)$ in $\mathbb{K}$ such that $\sum_{i\in\mathbb{Z}_{>0}} a_ib_i=v$.

A <u>topological $\mathbb{K}$-vector space</u> is a $\mathbb{K}$-vector space V with a topology such that

$$\begin{matrix} V \times V \to V \\ (v_1, v_2) \mapsto v_1 + v_2 \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{K} \times V \to V \\ (c, v) \mapsto cv \end{matrix}$$

are continuous.

A <u>topological field</u> is a field $\mathbb{K}$ with a topology such that

$$\begin{matrix} \mathbb{K} \times \mathbb{K} \to \mathbb{K} \\ (c_1, c_2) \mapsto c_1 + c_2 \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{K} \times \mathbb{K} \to \mathbb{K} \\ (c_1, c_2) \mapsto (c_1 c_2) \end{matrix}$$

are continuous.

<u>Theorem</u> Let $(H, \langle\,\rangle)$ be a Hilbert space. If H has a countable dense set then

$$H \cong \ell^2.$$

<u>Proof</u> Assume $(v_1, v_2, \ldots)$ is a countable dense set. Let $(b_1, b_2, \ldots)$ be a subset of $\{v_1, v_2, \ldots\}$ which is linearly independent.

Use $(b_1, b_2, \ldots)$ to produce an orthonormal sequence $(a_1, a_2, \ldots)$ by using Gram-Schmidt

$a_1 = b_1$, $a_{n+1} = \dfrac{b_{n+1} - \langle b_{n+1}, a_1\rangle a_1 - \cdots - \langle b_{n+1}, a_n\rangle a_n}{\| b_{n+1} - \langle b_{n+1}, a_1\rangle a_1 - \cdots - \langle b_{n+1}, a_n\rangle a_n \|}$

Let $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, ...

and define a linear transformation

$$\begin{aligned} \Phi : H &\to \ell^2 \\ a_i &\mapsto e_i. \end{aligned}$$

so that $\Phi(c_1 a_1 + c_2 a_2 + \cdots) = c_1 e_1 + c_2 e_2 + \cdots$.

To show: (a) Φ is a function

(b) If $x, y \in H$ then $\langle \Phi(x), \Phi(y)\rangle = \langle x, y\rangle$.

(c) Φ is bijective.

(a) To show: If $x \in H$ then there is a unique $\Phi(x)$ and $\Phi(x) \in \ell^2$.

Assume $x \in H$

Let $B' = \{v_1, v_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$

Then $B \subseteq B' \subseteq \mathrm{span}(B)$

Since $H = \overline{B'}$ then $H = \overline{B'} \subseteq \overline{\mathrm{span}(B)}$

So $H = \overline{\mathrm{span}(B)}$.

So $x = \sum_{n=1}^{\infty} \langle x, a_n\rangle a_n = \lim_{k\to\infty}\left(\sum_{n=1}^{k} \langle x, a_n\rangle a_n\right).$

Is there a step missing here?

So $\Phi(x) = (\langle x, a_1\rangle, \langle x, a_2\rangle, \ldots)$.

(b) Assume $x, y \in H$.

To show: $\langle \Phi(x), \Phi(y)\rangle = \langle x, y\rangle$.

$$\langle x, y\rangle = \left\langle \sum_{n=1}^{\infty} \langle x, a_n\rangle a_n, \sum_{m=1}^{\infty} \langle y, a_m\rangle a_m \right\rangle$$

$$= \sum_{n=1}^{\infty} \langle x, a_n\rangle \langle y, a_n\rangle$$

$$= \langle (\langle x, a_1\rangle, \langle x, a_2\rangle, \ldots), (\langle y, a_1\rangle, \langle y, a_2\rangle, \ldots)\rangle$$

$$= \langle \Phi(x), \Phi(y)\rangle.$$

(c) To show: (ca) Φ is injective

(cb) Φ is surjective.

(ca) Since $\langle \Phi(x), \Phi(x)\rangle = \langle x, x\rangle$ then

$\|x\| = \|\Phi(x)\|_2$ and $d_H(x,y) = d_{\ell^2}(\Phi(x), \Phi(y))$.

To show: If $x, y \in H$ and $\Phi(x) = \Phi(y)$ then $x = y$.

Assume $x, y \in H$ and $\Phi(x) = \Phi(y)$.

To show: $x = y$.

To show: $d_H(x,y) = 0$.

$d_H(x,y) = d_{\ell^2}(\Phi(x), \Phi(y)) = d_{\ell^2}(\Phi(x), \Phi(x)) = 0$.

So $x = y$ and Φ is injective.

kb) ~~To show~~: Φ is surjective.

To show: If $c = (c_1, c_2, \ldots) \in \ell^2$ then there exists $x \in H$ such that $\Phi(x) = c$.

Assume $c = (c_1, c_2, \ldots) \in \ell^2$.

To show: There exists $x \in H$ such that $\Phi(x) = c$.

Let $x = c_1 a_1 + c_2 a_2 + \cdots$.

Since H is complete then

$$x = \lim_{k \to \infty} \left(\sum_{n=1}^{k} c_n a_n \right) \text{ is in } H.$$

Is there a step missing here?

Then $\Phi(x) = (c_1, c_2, \ldots) = c$.

So Φ is surjective. //