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Metric and Hilbert Spaces : Lecture 16

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(1)

Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$.

The <u>boundary</u> of A is $\partial A = \overline{A} \cap \overline{A^c}$

The set A is <u>dense</u> in X if $\overline{A} = X$

The set A is <u>nowhere dense in X</u> if $(\overline{A})^\circ = \emptyset$.

<u>Examples</u>: (1) In $\mathbb{R}$ $\partial([0,1)) = \{0,1\}$.

(2) $\mathbb{Q}$ is dense in $\mathbb{R}$.

(3) $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$

<u>Exercise</u> Let (X,d) be a metric space and let $U \subseteq X$ and $V \subseteq X$. Show that if U and V are open dense in X then $U \cap V$ is open and dense in X.

<u>Proof</u>

Assume U and V are open dense in X.

Since finite intersections of open sets are open then $U \cap V$ is open in X.

To show: $\overline{U \cap V} = X$.

Let $x \in X$.

To show: x is a close point to $U \cap V$.

Let $N \in \mathcal{N}(x)$.

To show: $N \cap (U \cap V) \neq \emptyset$

We know $N \cap U \neq \emptyset$ and $N \cap V \neq \emptyset$ since $\overline{U} = X$ and $\overline{V} = X$.

Let $y \in N \cap U$.

Then $N \cap U$ is a neighborhood of y.

Since $\overline{V} = X$ then $(N \cap U) \cap V \neq \emptyset$. //.

<u>Baire theorem</u> Let (X, d) be a <u>complete</u> metric space. Let $U_1, U_2, \ldots$ be open dense subsets of X. Show that $(U_1 \cap U_2 \cap \cdots)$ is dense in X.

<u>Example</u> Let X be a complete normed vector space over $\mathbb{R}$. A <u>sphere</u> in X is

$$S_r(a) = \{x \in X \mid d(x,a) = r\} \text{ for } a \in X, r \in \mathbb{R}_{>0}.$$

(a) $S_r(a)$ is nowhere dense

(b) There does not exist $r_1, r_2, \ldots \in \mathbb{R}_{>0}$ and $a_1, a_2, \ldots \in X$ such that

$$(S_{r_1}(a_1) \cup S_{r_2}(a_2) \cup \cdots) = X$$

(c) If $X = \mathbb{Z}$ then there does exist $r_1, r_2, \ldots \in \mathbb{R}_{>0}$ and $a_1, a_2, \ldots \in \mathbb{Z}$ such that

$$(S_{r_1}(a_1) \cup S_{r_2}(a_2) \cup \cdots) = \mathbb{Z},$$

namely

$$S_1(0) \cup S_2(0) \cup \cdots = \mathbb{Z}$$

since $S_\ell(0) = \{-\ell, 0, \ell\}$

Note that $S_1(0)$ is <u>not</u> nowhere dense in $\mathbb{Z}$

Baire Theorem (nowhere dense version)

Let (X,d) be a complete metric space.
Let $F_1, F_2, \ldots$ be nowhere dense subsets of X.
Then $(\overline{F_1 \cup F_2 \cup \cdots})^\circ \neq \phi$

Vector spaces

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

A normed vector space is a vector space V with a function $\| \ \| : V \to \mathbb{R}_{\geq 0}$ such that

(a) If $x, y \in V$ then $\|x+y\| \leq \|x\| + \|y\|$
(b) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c| \cdot \|v\|$.
(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.

Let $(V, \| \ \|)$ be a normed vector space.
The norm metric on V is the function

$$d : V \times V \to \mathbb{R}_{\geq 0} \text{ given by } d(x,y) = \|y-x\|.$$

Proposition Let $(V, \| \ \|)$ an normed vector space.
Let $d : V \times V \to \mathbb{R}_{\geq 0}$ be the norm metric on V.
Then d is a metric on V.

Proof To show: (a) If $v \in V$ then $d(v,v) = 0$.

(b) If $v_1, v_2 \in V$ and $d(v_1, v_2) = 0$ then $v_1 = v_2$.

(c) If $v_1, v_2 \in V$ then $d(v_1, v_2) = d(v_2, v_1)$

(d) If $x, y, z \in V$ then

$$d(x,y) \le d(x,z) + d(z,y).$$

(a) Assume $v \in V$.

Then

$$d(v,v) = \|v - v\| = \|0\| = |0| \cdot \|0\| = 0 \cdot \|0\| = 0.$$

(b) Assume $v_1, v_2 \in V$ and $d(v_1, v_2) = 0$.

To show: $v_1 = v_2$.

Since $\|v_1 - v_2\| = d(v_1, v_2) = 0$ then $v_1 - v_2 = 0$.

So $v_1 = v_2$

(c) Assume $v_1, v_2 \in V$.

Then

$$\begin{aligned} d(v_1, v_2) &= \|v_2 - v_1\| = \|(-1)(v_1 - v_2)\| \\ &= |-1| \cdot \|v_1 - v_2\| = 1 \cdot \|v_1 - v_2\| \\ &= \|v_1 - v_2\| = d(v_2, v_1). \end{aligned}$$

(d) Assume $x, y, z \in V$.

To show: $d(x,y) \le d(x,z) + d(z,y)$.

$$d(x,y) = \|y-x\| = \| y-z+z-x\|$$
$$\le \|y-z\| + \|z-x\| = d(z,y) + d(x,z). //.$$

A <u>Banach space</u> is a complete normed vector space $(V, \| \|)$ such that if d is the norm metric then (V,d) is Cauchy compact.).