Metric and Hilbert Spaces Lect. 12

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The real numbers

$$\mathbb{R} = \left\{ \pm\left(a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-(\ell+1)}\left(\tfrac{1}{10}\right)^{-\ell+1} + \cdots\right) \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z} \\ a_i \in \{0,1,\ldots,9\} \end{array} \right\}$$

$$\cup|$$

$$\mathbb{R}_{\geq 0} = \left\{ a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-(\ell+1)}\left(\tfrac{1}{10}\right)^{-\ell+1} + \cdots \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z} \\ a_i \in \{0,1,\ldots,9\} \end{array} \right\}.$$

The order on $\mathbb{R}$

Define $x \leq y$ if there exists $z \in \mathbb{R}_{\geq 0}$ such that $x + z = y$.

Prove the following:

(1) If $a,b,c \in \mathbb{R}$ and $a \leq b$ then $a+c \leq b+c$

(2) If $a,b \in \mathbb{R}_{\geq 0}$ and then $ab \in \mathbb{R}_{\geq 0}$.

(3) If $a \in \mathbb{R}$ then $a \leq a$

(4) If $a,b,c \in \mathbb{R}$ and $a \leq b$ and $b \leq c$ then $a \leq c$

(5) If $a,b \in \mathbb{R}$ and $x \leq y$ and $y \leq x$ then $x = y$.

(6) If $x,y \in \mathbb{R}$ then $x \leq y$ or $y \leq x$.

Archimedes' axiom If $x,y \in \mathbb{R}_{\geq 0}$ and $x > 0$ then there exists $n \in \mathbb{Z}_{>0}$ such that $nx > y$.

Least upper bound property If $A \subseteq \mathbb{R}_{\geq 0}$ and $A \neq \emptyset$ and A is bounded then $\sup(A)$ exists in $\mathbb{R}_{\geq 0}$.

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The metric on $\mathbb{R}$

Define $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$d(x,y) = |y-x|, \quad \text{where}$$

$|y-x| \in \mathbb{R}_{\geq 0}$ such that $x + |y-x| = y$ or $y + |y-x| = x$.

Prove the following:

(a) If $x \in \mathbb{R}$ then $d(x,x) = 0$.

(b) If $x, y \in \mathbb{R}$ and $d(x,y) = 0$ then $x = y$.

(c) If $x, y \in \mathbb{R}$ then $d(x,y) = d(y,x)$.

(d) If $x, y, z \in \mathbb{R}$ then $d(x,y) \leq d(x,z) + d(z,y)$.

(e) If $x, y \in \mathbb{R}$ then $|y-x|$ exists and is unique.

These properties make $\mathbb{R}$ into a metric space, and thus a uniform space and a topological space.

Theorem (a) $\mathbb{R}_{\geq 0}$ is complete.
(b) $\mathbb{R}_{\geq 0}$ is locally compact.

A *locally compact space* is a topological space $(X, \mathcal{T})$ such that X is Hausdorff and if $x \in X$ then there exists $N \in \mathcal{N}(x)$ such that N is cover compact.

18.08.2017 3
A. Ram

Sketches of proofs:

(1) Archimedes axiom.

Let $x = 3781.2469572$

$y = 15827631.399402876.$

Find $n \in \mathbb{Z}_{>0}$ such that $y < nx$.

Let $n = 10^m$ where $m = \max\{0, k-l\} + 52$

where k is the first nonzero decimal place of y

l is the first nonzer decimal place of x.

To show: $y < nx$.

(2) Let

$A = \{1, 1.01, 1.01001, 1.010010001, \ldots\}$

Let $z = 1.010010001\ldots$

so that $z_{\le k} = \max(A_{\le k})$ where

$A_{\le k} = \{a_{\le k} \mid a \in A\}$.

Note that $A_{\le k}$ is a finite set, so it has a maximum.

To show: $z = \sup(A)$.

(3) Let $(x^{(1)}, x^{(2)}, \dots)$ be a Cauchy sequence in $\mathbb{R}_{\geq 0}$.

Find $z = \lim_{k\to\infty} x^{(k)}$.

Let $z_{\leq k} = x^{(\ell_k)}_{\leq k}$ where

$\ell_k \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq \ell_k}$ then

$$d(x^{(m)}, x^{(n)}) \leq \left(\frac{1}{10}\right)^{k+61}.$$

Then z is a real number (all its decimal digits are determined).

To show: $z = \lim_{k\to\infty} x^{(k)}$.