Metric and Hilbert Spaces Lecture 10

Let $(X,\mathcal{T})$ be a topological space. Let $A \subseteq X$.
The set A is <u>cover compact</u> if A satisfies:
If $\mathcal{S} \subseteq \mathcal{T}$ and $\left(\bigcup_{U\in\mathcal{S}} U\right) \supseteq A$ then
there exists $\ell \in \mathbb{Z}_{>0}$ ~~such that~~ and $U_1, \ldots, U_\ell \in \mathcal{S}$
such that $U_1 \cup \cdots \cup U_\ell \supseteq A$.

In English: Every open cover has a finite subcover.

Let (X,d) be a metric space. Let $A \subseteq X$.
The set A is <u>ball compact</u> if A satisfies:
If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and
$x_1, x_2, \ldots, x_\ell \in X$ such that

$$B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell) \supseteq A.$$

Let (X,d) be a metric space. Let $\mathcal{T}$ be the metric space topology.
Let $\varepsilon \in \mathbb{R}_{>0}$. Then

$$\mathcal{S}_\varepsilon = \{B_\varepsilon(x) \mid x \in X\} \quad \text{satisfies}$$

$$\mathcal{S}_\varepsilon \subseteq \mathcal{T} \text{ and } \left(\bigcup_{B\in\mathcal{S}_\varepsilon} B\right) \supseteq A.$$

So $\mathcal{S}_\varepsilon$ is an open cover of A.

Let (X,d) be a metric space. Let $A \subseteq X$.

The set A is <u>bounded</u> if A satisfies:

there exists $x \in X$ and $M \in \mathbb{R}_{>0}$ such that
$$B_M(x) \supseteq A.$$

Alternatively, A is bounded if there exists $x \in X$ and $M \in \mathbb{R}_{>0}$ such that
if $a \in A$ then $d(a,x) < M$.

<u>Proposition</u> Let (X,d) be a metric space. Let $A \subseteq X$. ~~If~~ If A is ball compact then A is bounded.

<u>Proof</u> Assume A is ball compact.

To show: A is bounded.

To show: There exist $x \in X$ and $M \in \mathbb{Z}_{>0}$ such that $A \subseteq B_M(x)$.

Since A is ball compact there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in X$ such that
$$B_1(x_1) \cup B_1(x_2) \cup \cdots \cup B_1(x_\ell) \supseteq A$$
Let $x = x_1$ and $M = 100 + \max\{d(x_2,x_1), \ldots, d(x_\ell,x_1)\}$.

To show: $A \subseteq B_M(x)$.

To show: If $a \in A$ then $d(a,x) < M$.

Assume $a \in A$.

Since $B_1(x_1) \cup \cdots \cup B_1(x_\ell) \supseteq A$ then there exists $k \in \{1, \ldots, \ell\}$ such that $d(a, x_k) < 1$.

To show: $d(a,x) < M$.

$$\begin{aligned} d(a,x) &\le d(a,x_k) + d(x_k, x) \\ &< 1 + d(x_k, x) < M. \end{aligned}$$

So $A \subseteq B_M(x)$.

So A is bounded. //.

<u>Proposition</u> Let $A \subseteq \mathbb{R}$, where the metric on $\mathbb{R}$ is $d(x,y) = |y-x|$. If A is bounded then A is ball compact.

<u>Proof</u> Assume A is bounded.

To show: A is ball compact.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exist $x_1, x_2, \ldots, x_\ell \in \mathbb{R}$ such that $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell) \supseteq A$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Since A is bounded there exist $x \in X$ and $M \in \mathbb{R}_{>0}$ such that $B_M(x) \supseteq A$.

Let $\ell \in \mathbb{Z}_{>0}$ be minimal such that $\ell - 1 > \frac{2M}{\varepsilon}$.

Let
$$x_1 = x - M,\ x_2 = x_1 + \varepsilon,\ x_3 = x_1 + 2\varepsilon,\ \ldots,\ x_\ell = x_1 + (\ell-1)\varepsilon.$$

Since $x_\ell = x_1 + (\ell-1)\varepsilon = x - M + (\ell-1)\varepsilon > x - M + 2M$, then

$$\begin{aligned}
&B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell) \\
&\quad = (x_1 - \varepsilon, x_1 + \varepsilon) \cup (x_2 - \varepsilon, x_2 + \varepsilon) \cup \cdots \cup (x_\ell - \varepsilon, x_\ell + \varepsilon) \\
&\quad = (x_1 - \varepsilon, x_1 + \varepsilon) \cup (x_1, x_1 + 2\varepsilon) \cup \cdots \cup (x_1 + (\ell-2)\varepsilon, x_1 + \ell\varepsilon) \\
&\quad \supseteq (x - M, x + M) = B_M(x) \supseteq A
\end{aligned}$$

So A is ball compact. //