16 a) Assume $(b_1, b_2, \ldots)$ is a Schauder basis of $V$.

To show: $\overline{\mathbb{K}\text{-span}\{b_1, b_2, \ldots\}} = V$.

To show: If $v \in V$ then $v \in \overline{\mathbb{K}\text{-span}\{b_1, b_2, \ldots\}}$.

Assume $v \in V$.

Then there exists a unique sequence $(a_1, a_2, \ldots)$ in $\mathbb{K}$ such that $v = \sum_{i=1}^{\infty} a_i b_i$.

So $v = \lim_{n \to \infty} s_n$, where $s_n = a_1 b_1 + \cdots + a_n b_n$.

Since $s_n \in \mathbb{K}\text{-span}\{b_1, b_2, \ldots\}$ and

$v = \lim_{n \to \infty} s_n$ then $v \in \overline{\mathbb{K}\text{-span}\{b_1, b_2, \ldots\}}$

So $V \subseteq \overline{\mathbb{K}\text{-span}\{b_1, b_2, \ldots\}}$

So $V = \overline{\mathbb{K}\text{-span}\{b_1, b_2, \ldots\}}$

~~So // V is a total~~

So $\{b_1, b_2, \ldots\}$ is a total set.

1(b) Assume $(b_1, b_2, \ldots)$ is a Schauder basis of $V$.

To show: $V$ has a countable dense set.

Let
$$\mathbb{L} = \begin{cases} \mathbb{Q}, & \text{if } \mathbb{K} = \mathbb{R}, \\ \mathbb{Q} + i\mathbb{Q}, & \text{if } \mathbb{K} = \mathbb{C}, \end{cases}$$

so that $\mathbb{L}$ is countable and $\overline{\mathbb{L}} = \mathbb{K}$.

Then $\mathbb{L}\text{-span}\{b_1, b_2, \ldots\}$ is countable and

$$\overline{\mathbb{L}\text{-span}\{b_1, b_2, \ldots\}} = \overline{\mathbb{K}\text{-span}\{b_1, b_2, \ldots\}}.$$

By part (a), $\overline{\mathbb{K}\text{-span}\{b_1, b_2, \ldots\}} = V$.

So $\mathbb{L}\text{-span}\{b_1, b_2, \ldots\}$ is countable and

$$\overline{\mathbb{L}\text{-span}\{b_1, b_2, \ldots\}} = V.$$

So $V$ has a countable dense set.

16c) To show: $(e_1, e_2, \ldots)$ is a Schauder basis of $\ell^p$

To show: If $v \in \ell^p$ then there exists a unique sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}$ such that $v = \sum_{i=1}^{\infty} a_i e_i$.

Assume $v = (a_1, a_2, \ldots)$ is in $\ell^p$.

Claim: $v = \sum_{i=1}^{\infty} a_i e_i$.

Let $s_n = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n = (a_1, a_2, \ldots, a_n, 0, \ldots)$

To show: $\lim_{n \to \infty} s_n = v$.

To show: $\lim_{n \to \infty} \|v - s_n\|_p = 0$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq \ell}$ then $\|v - s_n\|_p < \varepsilon$.

We know that $\lim_{n \to \infty} \|s_n\|_p = \|v\|_p$, by definition of $\|v\|_p$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Then there exists $\ell \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq \ell}$ then $\big| \|v\|_p - \|s_n\|_p \big| < \varepsilon$.

To show: ~~If~~ $n \in \mathbb{Z}_{\geq \ell}$ then $\|v - s_n\|_p < \varepsilon$.

Since $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and
$x \mapsto x^p$

$\lim_{n\to\infty} \|s_n\|_p = \|v\|_p$ then $\lim_{n\to\infty} \|s_n\|_p^p = \|v\|_p^p$.

So there exist $l \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq l}$ then $\left| \|v\|_p^p - \|s_n\|_p^p \right| < \varepsilon^p$.

To show: $\|v - s_n\|_p < \varepsilon$.

To show: $\|v - s_n\|_p^p < \varepsilon^p$.

$$\|v - s_n\|_p^p = \sum_{j=n+1}^{\infty} |a_j|^p = \|v\|_p^p - \|s_n\|_p^p < \varepsilon^p.$$

So $\|v - s_n\|_p < \varepsilon$.

So $\cancel{\lim_{n\to\infty} \|s_n\|_p = \|}$ $\lim_{n\to\infty} s_n = v$ in $\ell^p$.

<u>Now show</u>: That the expansion is unique.

Assume $(a_1, a_2, \ldots) \neq (b_1, b_2, \ldots)$

To show: $\sum_{i=1}^{\infty} a_i e_i \neq \sum_{i=1}^{\infty} b_i e_i$ in $\ell^p$.

Let $j \in \mathbb{Z}_{>0}$ be minimal such that $b_j \neq a_j$.

Then

$$\|(b_1, b_2, \ldots) - (a_1, a_2, \ldots)\|_p$$

$$= \|(0, 0, \ldots, 0, b_j - a_j, \ldots)\|_p$$

$$\geq \left(|b_j - a_j|^p\right)^{1/p} = |b_j - a_j| > 0.$$

So $(b_1, b_2, \ldots) \neq (a_1, a_2, \ldots)$ in $\ell^p$.

(6d) Let $v = (1, 1, \ldots)$

Since $\sup\{1, 1, 1, \ldots\} = 1$ then $\|v\|_\infty = 1$.

So $v \in \ell^\infty$.

Let $e_i = (0, 0, \ldots, 0, \overset{i\text{th}}{1}, 0, 0, \ldots)$ and

$s_1 = e_1, \quad s_2 = e_1 + e_2, \ldots$

Then $\|v - s_n\|_\infty = \sup\{0, \ldots, 0, 1, 1, \ldots\} = 1$

So $\lim\limits_{n \to \infty} \|v - s_n\|_\infty \neq 0.$

So $v \neq \sum\limits_{i=1}^{\infty} e_i$ in $\ell^\infty$.