(3g) $\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \{0, 1, \ldots, p-1\}\}$

$= \{x \in \mathbb{Q}_p \mid |x|_p < e^1\}$

$= B_e(0)$

the ball of radius e centred at 0.

So $\mathbb{Z}_p$ is open (it is an open ball).

(3f) To show: (fa) $\mathbb{Z}_p$ is ball compact

(fb) $\mathbb{Z}_p$ is Cauchy compact.

(fa) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists a finite set $L \subseteq \mathbb{Z}_p$ such that $\mathbb{Z}_p \subseteq \bigcup_{x \in L} B_\varepsilon(x)$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Let $l \in \mathbb{Z}_{>0}$ such that $e^{-(l-1)} \geq \varepsilon > e^{-l}$.

Let $X = \{a_0 + a_1 p + \cdots + a_l p^l \mid a_i \in \{0, 1, \ldots, p-1\}$

Since $\mathrm{Card}(X) = (p-1)^{l+1}$ then X is finite.

Then $B_\varepsilon(a_0 + a_1 + \cdots + a_l p^l)$

$= \{a_0 + a_1 + \cdots + a_l p^l + b_{l+1} p^{l+1} + \cdots \mid b_i \in \{0, \ldots, p-1\}\}$

So $\mathbb{Z}_p = \bigcup_{x \in X} B_\varepsilon(x)$.

So $\mathbb{Z}_p$ is a finite union of ε-balls.

So $\mathbb{Z}_p$ is ball compact.

(4b) To show: $\mathbb{Z}_p$ is Cauchy compact.

To show: If $(z_1, z_2, \ldots)$ is a Cauchy sequence in $\mathbb{Z}_p$ then there exists $z \in \mathbb{Z}_p$ such that $\lim_{n\to\infty} z_n = z$.

$$z_1 = a_{10} + a_{11}p + a_{12}p^2 + \cdots$$
$$z_2 = a_{20} + a_{21}p + a_{22}p^2 + \cdots$$
$$\vdots$$

Let $k \in \mathbb{Z}_{\geq 0}$. Since $(z_1, z_2, \ldots)$ is a Cauchy sequence then there exists $N_k \in \mathbb{Z}_{>0}$ such that

if $m, n \in \mathbb{Z}_{\geq N_k}$ then $d(z_m, z_n) \leq e^{-k}$.

So $a_{mk} = a_{nk}$ for $m, n \in \mathbb{Z}_{\geq N_k}$.

Let $a_k = a_{nk}$, where $n \in \mathbb{Z}_{\geq N_k}$.

Let $z = a_0 + a_1 p + a_2 p + \cdots$

To show: $\lim_{k\to\infty} z_k = z$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that $d(z_k, z) < \varepsilon$ if $k \in \mathbb{Z}_{\geq N}$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that $d(z_k, z) < \varepsilon$.

Let $\ell \in \mathbb{Z}_{>0}$ such that $e^{-(\ell+1)} < \varepsilon \leq e^{-\ell}$.

Let $N = N_\ell$ where N_ℓ is defined as above so that if $m, n \in \mathbb{Z}_{\geq \ell}$ then $a_{m\ell} = a_{n\ell} = a_\ell$.

Thus, if $k \in \mathbb{Z}_{\geq N}$ then $a_0 = a_{k0}, a_1 = a_{k1}, \ldots, a_\ell = a_{k\ell}$.

So $|z - z_k|_p < e^{-(\ell+1)} < \varepsilon$.

So $d(z_k, z) < \varepsilon$.

So $\lim_{k\to\infty} z_k = z$.

So $\mathbb{Z}_p$ is Cauchy compact.

Since $\mathbb{Z}_p$ is ball compact ~~then~~ and Cauchy compact then $\mathbb{Z}_p$ is ball compact. //