(9a) To construct the Cantor set begin with

$A = [0,1] = \{ a_1(\tfrac{1}{3})^1 + a_2(\tfrac{1}{3})^2 + \dots \mid a_i \in \{0,1,2\} \}$.

Removing the middle third to get

$A_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$

is the same as removing elements of A with $a_1 = 1$.

Removing the middle third of each of the components of A_1 is the same as removing elements of A with $a_2 = 1$.

The process continues and so the Cantor set $C = \{ a_1(\tfrac{1}{3})^1 + a_2(\tfrac{1}{3})^2 + \cdots \mid a_i \in \{0,2\} \}$.

(9b) Since $C \subseteq \mathbb{R}$ then $\mathrm{Card}(C) \leq \mathrm{Card}(\mathbb{R})$.

To show: $\mathrm{Card}(C) = \mathrm{Card}(\mathbb{R})$

To show: C is not countable.

Proof by contradiction.

Assume $C = \{c_1, c_2, \dots\}$ with

$c_1 = a_{11}(\tfrac{1}{3})^1 + a_{12}(\tfrac{1}{3})^2 + \dots$

$c_2 = a_{21}(\tfrac{1}{3})^1 + a_{22}(\tfrac{1}{3})^2 + \dots$

$\vdots$

where $a_{ij} \in \{0,2\}$.

Let $c = a_1(\frac{1}{3})^1 + a_2(\frac{1}{3})^2 + \cdots$

where $a_j = \begin{cases} 0, & \text{if } a_{jj} = 2 \\ 2, & \text{if } a_{jj} = 0 \end{cases}$

Then c is a $(\frac{1}{3})$-adic expansion with coefficients in $\{0, 2\}$ and

$$c \notin \{c_1, c_2, \ldots\}$$

since c differs from c_j at the coefficient of $(\frac{1}{3})^j$.

So C is not countable.

So $\mathrm{Card}(C) = \mathrm{Card}(\mathbb{R})$.

(9c) Let $x = x_1(\frac{1}{3})^1 + x_2(\frac{1}{3})^2 + \cdots \in C$ so that $x_j \in \{0, 2\}$.

Let $\varepsilon \in \mathbb{R}_{>0}$

Let $k \in \mathbb{Z}_{>0}$ with $(\frac{1}{3})^k < \varepsilon$.

Let $y = y_1(\frac{1}{3})^1 + y_2(\frac{1}{3})^2 + \cdots$ with

$y_{k+1} = \begin{cases} 2, & \text{if } x_{k+1} = 0 \\ 0, & \text{if } x_{k+1} = 2 \end{cases}$ and $y_j = x_j$ if $j \neq k+1$.

So y differs from x only at the coefficient of $(\frac{1}{3})^{k+1}$. So $|y - x| < (\frac{1}{3})^k < \varepsilon$.

So $y \in C$, $y \in (x-\varepsilon, x+\varepsilon)$ and $y \neq x$. So $(x-\varepsilon, x+\varepsilon) \cap C \neq \{x\}$

(9e) Let C be the Cantor set, $C \subseteq [0,1]$.

Since $C = \left(\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \cdots \right)^c$ then

C is the complement of a union of open intervals.

So C is the complement of an open set.

So C is closed.

(9f) Since $C \subseteq [0,1]$ then C is bounded (if $x, y \in C$ then $d(x,y) \le 1 < 2$).

So C is closed and bounded subset of $\mathbb{R}$

Thus by Chapter 6 of the notes, Theorem 6.0.2

C is compact.

(9d) To show: C is totally disconnected

Let $x, y \in C$ with $x \neq y$. Assume $x < y$.

To show: There does not exist a connected subset ~~of~~ E containing x and y.

Let E be a subset of C containing x and y.

Let $N \in \mathbb{Z}_{>0}$ with $\frac{1}{3^N} < \frac{y-x}{3}$ and let $k \in \mathbb{Z}_{>0}$ be the smallest positive integer such that

$x < \frac{2k+1}{3^N}$. Then $\frac{2k+2}{3^N} < y$.

Let

$A = (-\infty, \frac{2k+1}{3^N}) \cap C$ and $B = (\frac{2k+2}{3^N}, \infty) \cap C$

Then $x \notin A$ and $y \in B$ and $A \cap B = \phi$.

Since

$(\frac{2k+1}{3^N}, \frac{2k+2}{3^N}) \subseteq C^c$ then $E \subseteq A \cup B$.

So E is not connected.

So there does not exist a connected set containing x and y.

So each connected component of C contains only a single element.

So C is totally disconnected.