(8a) Let (X,d) be a complete metric space and let $f: X \to X$ be a contraction mapping.
Let $x \in X$ and let $(x_1, x_2, \ldots)$ be the sequence in X given by

$$x_1 = f(x), \quad x_2 = f(f(x)), \quad x_3 = f(f(f(x))), \ldots$$

To show: (aa) $(x_1, x_2, \ldots)$ converges.

(ab) If $p = \lim_{n\to\infty} x_n$ then $f(p) = p$.

(ac) If $q \in X$ and $f(q) = q$ then $p = q$.

(aa) To show: $(x_1, x_2, \ldots)$ converges

Since X is complete, To show: $(x_1, x_2, \ldots)$ is Cauchy.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq N}$ then $d(x_m, x_n) < \varepsilon$.

Let $\alpha \in \mathbb{R}_{(0,1)}$ such that

if $x, y \in X$ then $d(f(x), f(y)) \leq \alpha\, d(x,y)$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let N be the smallest integer in $\mathbb{Z}_{>0}$ such that

$$\frac{\alpha^N d(f(x), x)}{1-\alpha} < \varepsilon \qquad \left(\text{so } \alpha^N < \frac{\varepsilon(1-\alpha)}{d(f(x), x)} \right)$$

To show: If $m, n \in \mathbb{Z}_{\geq N}$ then $d(x_m, x_n) < \varepsilon$

Assume $m, n \in \mathbb{Z}_{\geq N}$

Assume $n < m$.

To show: $d(x_m, x_n) < \varepsilon$.

Since

$$d(x_2, x_1) = d(f^2(x), f(x)) \le \alpha\, d(f(x), x),$$

$$d(x_3, x_2) = d(f^3(x), f^2(x)) \le \alpha\, d(f^2(x), f(x)) \le \alpha^2 d(f(x), x),$$

$$d(x_4, x_3) = d(f^4(x), f^3(x)) \le \alpha\, d(f^3(x), f^2(x)) \le \alpha^3 d(f(x), x),$$

$$\vdots$$

then

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\le \alpha^n d(f(x), x) + \alpha^{n+1} d(f(x), x) + \cdots + \alpha^{m-1} d(f(x), x) \\ &= (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1})\, d(f(x), x) \\ &\le \alpha^n (1 + \alpha + \alpha^2 + \cdots)\, d(f(x), x) = \frac{\alpha^n}{1-\alpha} d(f(x), x) \\ &< \varepsilon. \end{aligned}$$

So $(x_1, x_2, \ldots)$ is a Cauchy sequence in X.

Since X is complete then $(x_1, x_2, \ldots)$ converges.

(ab) Assume $p = \lim_{n \to \infty} x_n$.

To show: $f(p) = p$.

To show: $d(f(p), p) = 0$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then $d(f(p), p) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Since $\lim_{n\to\infty} x_n = p$ there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, p) < \frac{\varepsilon}{2}$

So
$$\begin{aligned} d(f(p),p) &\leq d(f(p), x_{N+1}) + d(x_{N+1}, p) \\ &\leq \alpha\, d(p, x_N) + d(x_{N+1}, p) \qquad \left(\text{since } x_{N+1} = f(x_N)\right) \\ &< \alpha \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

So $d(f(p),p) = 0$.

So $f(p) = p$.

(ac) To show: If $q \in X$ and $f(q) = q$ then $q = p$.

Assume $q \in X$ and $f(q) = q$.

To show: $q = p$.

To show: $d(q,p) = 0$.

$$\begin{aligned} d(q,p) &= d(f(q), f(p)) \qquad \left(\text{since } q = f(q) \text{ and } p = f(p)\right) \\ &\leq \alpha\, d(q,p). \end{aligned}$$

So $(1-\alpha)\, d(q,p) \leq 0$.

Since $(1-\alpha)\, d(q,p) \geq 0$ and $(1-\alpha)\, d(q,p) \leq 0$

then $(1-\alpha)\, d(q,p) = 0$.

Since $\alpha \in \mathbb{R}_{(0,1)}$ then $1-\alpha \neq 0$ and so $d(q,p) = 0$.

So $p = q$. //

(8b) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function.

Let $a_1 \in \mathbb{R}$ and let $a_{n+1} = f(a_n)$.

Assume $(a_1, a_2, \ldots)$ converges and $a = \lim_{n\to\infty} a_n$.

To show: $f(a) = a$.

$$f(a) = f\left(\lim_{n\to\infty} a_n\right)$$

$$= \lim_{n\to\infty} f(a_n) \qquad \text{(since } f \text{ is continuous)}$$

$$= \lim_{n\to\infty} a_{n+1} = a.$$

(8c) If $f(x) = \frac{1}{x^2+1}$ then

$x = f(x)$ is $x = \frac{1}{x^2+1}$, which is $x^3 + x = 1$

which is $x^3 + x - 1 = 0$ (Thus the question sheet has a typographical error).

Let $a_1 = \frac{1}{2}$. Then

$$a_2 = f(a_1) = \frac{1}{\left(\frac{1}{2}\right)^2+1} = \frac{4}{5} = 0.8.$$

$$a_3 = f(a_2) = \frac{1}{\left(\frac{4}{5}\right)^2+1} = \frac{25}{41} \approx 0.609760976097\ldots$$

$$a_4 = f(a_3) = \frac{1681}{2306} \approx .728967.$$

$$a_5 = \frac{1}{a_4^2+1} \approx 0.6530046$$

$$a_6 = \frac{1}{a_5^2+1} \approx 0.7010582$$

$$a_7 = \frac{1}{a_6^2+1} \approx 0.6704737.$$

$$a_8 = \frac{1}{a_7^2+1} \approx 0.68987635$$

$$a_9 = \frac{1}{a_8^2+1} \approx 0.67753918$$

$$a_{10} = \frac{1}{a_9^2+1} \approx 0.68537308$$

$$a_{11} = \frac{1}{a_{10}^2+1} \approx 0.680394233$$

$$a_{12} = \frac{1}{a_{11}^2+1} \approx 0.6835567$$

$$a_{13} = \frac{1}{a_{12}^2+1} \approx 0.68154722$$

$$a_{14} = \frac{1}{a_{13}^2+1} \approx 0.68282382.$$

$$a_{15} = \frac{1}{a_{14}^2+1} \approx 0.6820126.$$

So, up to 3 decimal places of accuracy 0.682 should be a solution to $x^3+x-1=0$.

(8d) $x^3+x-1=0$ is $x=1-x^3$ which is

$x=f(x)$ with $f(x)=1-x^3$.

Let $a_1=\frac{1}{2}$

$a_2=f(a_1)=1-\left(\frac{1}{2}\right)^3=1-\frac{1}{8}=\frac{7}{8}=0.875$

$a_3\approx 0.330078$

$a_4\approx 0.964037$

$a_5\approx 0.104055$

If $0<a_i<\frac{1}{10^k}$ then a_{i+1} satisfies

$$1-\left(\frac{1}{10^k}\right)^3=1-\frac{1}{10^{3k}}<a_{i+1}<1-0^3=1.$$

If $1-\frac{1}{10^k}<a_i<1$ then a_{i+1} satisfies

$$0=1-1^3<a_{i+1}<1-\left(1-\frac{1}{10^k}\right)^3=1-1+\frac{3}{10^k}-\frac{3}{10^{2k}}+\frac{1}{10^{3k}}$$

$$=\frac{3}{10^k}-\frac{3}{10^{2k}}+\frac{1}{10^{3k}}<\frac{1}{10^{k-1}}$$

So $(a_1,a_2,\ldots)$ is a sequence with no cluster points but with limit points at 0 and 1.

(8e) Is $f(x) = 1 - x^3$ a contraction for $0 < x < 1$?

$$\frac{d(f(x), f(y))}{d(x,y)} = \frac{|(1-y^3) - (1-x^3)|}{|y-x|} = \frac{|y^3 - x^3|}{|y-x|}$$

$$= |y^2 + xy + x^2|$$

If $x = 1 - \frac{1}{10^2}$ and $y = 1 - \frac{1}{10^2}$ then

$$y^2 + xy + x^2 = 3\left(1 - \frac{1}{10^2}\right)^2 = 3\left(1 - \frac{2}{10^2} + \frac{1}{10^4}\right) > 3 - .06$$
$$= 2.94. > 1.$$

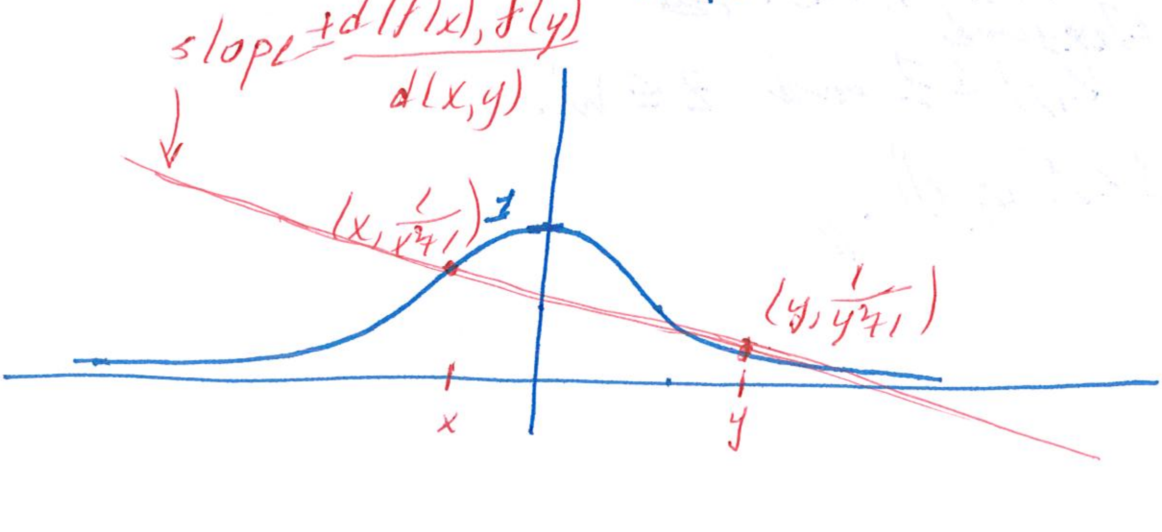
So there does not exist $\alpha \in \mathbb{R}_{(0,1)}$ with

$d(f(x), f(y)) < \alpha f(x,y)$ (with $x, y \in \mathbb{R}_{(0,1)}$).

So $f(x) = 1 - x^3$ is not a contraction.

Is $f(x) = \frac{1}{x^2+1}$ a contraction?

The graph of $y = \frac{1}{x^2+1}$ is



Then
$$\frac{d(f(x),f(y))}{d(x,y)} = \left| \frac{\frac{1}{x^2+1} - \frac{1}{y^2+1}}{y-x} \right|$$
is the absolute value of the slope of the line connecting $\left(x, \frac{1}{x^2+1}\right)$ and $\left(y, \frac{1}{y^2+1}\right)$.

This slope is bounded by the maximum of
$$\left| \frac{d}{dx}\left(\frac{1}{x^2+1}\right) \right| = \left| \frac{-2x}{(x^2+1)^3} \right|$$

The maximum of $\left|\frac{-2x}{(x^2+1)^2}\right|$ occurs at the critical points, i.e. at $x=a$ where
$$\frac{d}{dx}\left(\frac{-2x}{(x^2+1)^2}\right)\Bigg|_{x=a} = 0.$$

$$\frac{d}{dx}\left(\frac{-2x}{(x^2+1)^2}\right) = \frac{(-2)(2x)(-2x)}{(x^2+1)^3} + \frac{-2}{(x^2+1)^2}$$

$$= \frac{8x^2-2(x^2+1)}{(x^2+1)^3} = \frac{6x^2-2}{(x^2+1)^3} = \frac{6\left(x^2-\frac{1}{3}\right)}{(x^2+1)^3}.$$

So the critical points are at $x = \pm \frac{1}{\sqrt{3}}$

and the slopes at $x = \pm\frac{1}{\sqrt{3}}$ are

$$\frac{d}{dx}\left(\frac{1}{x^2+1}\right)\Bigg|_{x=\pm\frac{1}{\sqrt{3}}} = \frac{-2x}{(x^2+1)^2}\Bigg|_{x=\pm\frac{1}{\sqrt{3}}} = \frac{\mp\frac{2}{\sqrt{3}}}{\left(\frac{1}{3}+1\right)^2}$$

$$= \frac{\mp\frac{2}{\sqrt{3}}}{\frac{4^2}{3^2}} = \mp\frac{2\cdot 3^2}{4^2\sqrt{3}} = \mp\frac{3\sqrt{3}}{8}.$$

Thus

$$d(f(x), f(y)) \le \alpha\, d(x,y) \quad \text{with} \quad \alpha = \frac{3\sqrt{3}}{8}.$$

Since $\sqrt{3} < \sqrt{4} = 2$ then $\alpha < \frac{4}{8} = \frac{1}{2} < 1$

and so $f(x) = \frac{1}{x^2+1}$ is a contraction.

Thus $f(x)$ satisfies the hypotheses of the Banach fixed point theorem and the Banach fixed point theorem produces x such that $x = \frac{1}{x^2+1}$.