(7) Let $(X,\mathcal{T})$ be a topological space. ①
Let $E \subseteq X$.

(7a) To show: If E is path connected then E is connected.

To show: If E is not connected then E is not path connected.

Assume E is not connected.

To show: E is not path connected.

To show: There ~~exists~~ $x, y \in E$ with $x \neq y$ such that there does not exist a continuous function $f: \mathbb{R}_{[0,1]} \to E$ with $f(0) = x$ and $f(1) = y$.

Since E is not connected there exist $U \in \mathcal{T}$ and $V \in \mathcal{T}$ with

$E \cap U \neq \emptyset$, $V \cap E \neq \emptyset$, $(U \cup V) \supseteq E$, $(U \cap V) \cap E = \emptyset$.

Using that $E \cap U \neq \emptyset$ and $E \cap V \neq \emptyset$ let

$x \in E \cap U$ and $y = E \cap V$.

Since $(E \cap U) \cap (E \cap V) = \emptyset$ then $x \neq y$.

To show: there does not exist a continuous function $f: \mathbb{R}_{[0,1]} \to E$ with $f(0) = x$ and $f(1) = y$.

Assume $f : \mathbb{R}_{[0,1]} \to E$ is a function with

$f(0) = x$ and $f(1) = y$.

Let $\mathcal{T}_R$ be the standard topology on $\mathbb{R}_{[0,1]}$.

Let $A = f^{-1}(U \cap E)$ and $B = f^{-1}(V)$.

Then, since $x \in U$ and $y \in V$ then

$$0 \in A \text{ and } 1 \in B.$$

So $A \neq \phi$ and $B \neq \phi$.

Since $(U \cup V) \cap E = E$ then

$$A \cup B = f^{-1}(U \cup V \cap E) = f^{-1}(E) = \mathbb{R}_{[0,1]}.$$

Since $(U \cap V) \cap E = \phi$ then

$$A \cap B = \{x \in \mathbb{R}_{[0,1]} \mid f(x) \in U \text{ and } f(x) \in V\} = \phi$$

Since $\mathbb{R}_{[0,1]}$ is connected then $A \notin \mathcal{T}_R$ or $B \notin \mathcal{T}_R$.

So $f : \mathbb{R}_{[0,1]} \to E$ is not continuous.

So there exists $x, y \in E$ with $x \neq y$ such that there does not exist a continuous function $f : \mathbb{R}_{[0,1]} \to E$ with $f(0) = x$ and $f(1) = y$.

So E is not path connected.//

(7c) To show: (ca) If E is not connected then there exists a continuous surjective function $f: E \to \{0,1\}$

(cb) If there exists a continuous surjective function $f: E \to \{0,1\}$ then E is not connected.

(ca) Assume E is not connected.

So there exist $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_X$ such that

$$U \cap E \neq \emptyset, \quad V \cap E \neq \emptyset, \quad U \cup V \supseteq E, \quad (U \cap V) \cap E = \emptyset.$$

To show: there exists a continuous surjective function $f: E \to \{0,1\}$.

Define $f: E \to \{0,1\}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in U \cap E, \\ 1, & \text{if } x \in V \cap E. \end{cases}$$

Since $f^{-1}(\{0\}) = U \cap E$ and $U \in \mathcal{T}_X$

and $f^{-1}(\{1\}) = V \cap E$ and $V \in \mathcal{T}_X$

then f is continuous. (The subspace topology on E is $\mathcal{T}_E = \{U \cap E \mid U \in \mathcal{T}_X\}$).

Since $U \cap E \neq \emptyset$ and $V \cap E \neq \emptyset$ then f is surjective.

So f is a continuous surjective function.

(cb) Assume there exists a continuous surjective function $f: E \to \{0,1\}$

To show: E is not connected.

To show: There exist $U \in \mathcal{T}_E$ and $V \in \mathcal{T}_E$ such that

$$U \neq \phi, \quad V \neq \phi, \quad U \cup V = E \text{ and } U \cap V = \phi.$$

Let $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$.

Since f is continuous then $U \in \mathcal{T}_E$ and $V \in \mathcal{T}_E$

Since f is surjective then

$$U = f^{-1}(\{0\}) \neq \phi \text{ and } V = f^{-1}(\{1\}) \neq \phi.$$

Then
$$U \cup V = f^{-1}(\{0,1\}) = E \quad \text{and}$$
$$U \cap V = \{x \in E \mid f(x) \in \{0\} \text{ and } f(x) = \{1\}\} = \phi.$$

So E is not connected.

(7d) To show: If $\bar{E}$ is not connected then E is not connected.

Assume $\bar{E}$ is not connected.

To show: E is not connected

Let $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_X$ such that

$$U \cap \bar{E} \neq \phi, \quad V \cap \bar{E} \neq \phi, \quad U \cup V \supseteq \bar{E}, \quad (U \cap V) \cap \bar{E} = \phi.$$

To show: (da) $U \cap E \neq \phi$ and $V \cap E \neq \phi$

(db) $U \cup V \supseteq E$

(dc) $(U \cap V) \cap E = \phi$.

(db) Since $\bar{E} \supseteq E$ then $U \cup V \supseteq \bar{E} \supseteq E$.

(dc) Since $\bar{E} \supseteq E$ then $\phi = (U \cap V) \cap \bar{E} \supseteq (U \cap V) \cap E$.

(da) We know: $U \cap \bar{E} \neq \phi$.

Let $z \in U \cap \bar{E}$.

Since $z \in \bar{E}$ then z is a close point to E.

Since $z \in U$ then $U \in \mathcal{N}(z)$.

Since z is a close point to E then $U \cap E \neq \phi$.

We know: $V \cap \bar{E} \neq \phi$

Let $a \in V \cap \bar{E}$.

Since $a \in \bar{E}$ then a is a close point to E

Since $a \in V$ then $V \in \mathcal{N}(a)$

Since a is a close point to E then $V \cap E \neq \phi$.

So E is not connected.