(5a) The subsets of $X = \{1,2,3\}$ are

$$\begin{array}{ccc} & X & \\ \{1,2\} & \{1,3\} & \{2,3\} \\ \{1\} & \{2\} & \{3\} \\ & \phi & \end{array}$$

By renumbering the points, if $\mathcal{T}$ is a topology on X that contains a singleton set then the singleton set is $\{1\}$; if $\mathcal{T}$ contains two singleton sets then renumbering the points makes these $\{1\}$ and $\{2\}$. Thus, up to renumbering of the points in X, the topologies on X are (sets in $\mathcal{T}$ circled in red):

$$\begin{array}{ccc} & X & \\ \{1,2\} & \{1,3\} & \{2,3\} \\ \{1\} & \{2\} & \{3\} \\ & \phi & \end{array} \qquad \begin{array}{ccc} & X & \\ \{1,2\} & \{1,3\} & \{2,3\} \\ \{1\} & \{2\} & \{3\} \\ & \phi & \end{array}$$

$$\begin{array}{ccc} & X & \\ \{1,2\} & \{1,3\} & \{2,3\} \\ \{1\} & \{2\} & \{3\} \\ & \phi & \end{array} \xleftrightarrow{s_{1,3}} \begin{array}{ccc} & X & \\ \{1,2\} & \{1,3\} & \{2,3\} \\ \{1\} & \{2\} & \{3\} \\ & \phi & \end{array} \qquad \begin{array}{ccc} & X & \\ \{1,2\} & \{1,3\} & \{2,3\} \\ \{1\} & \{2\} & \{3\} \\ & \phi & \end{array}$$

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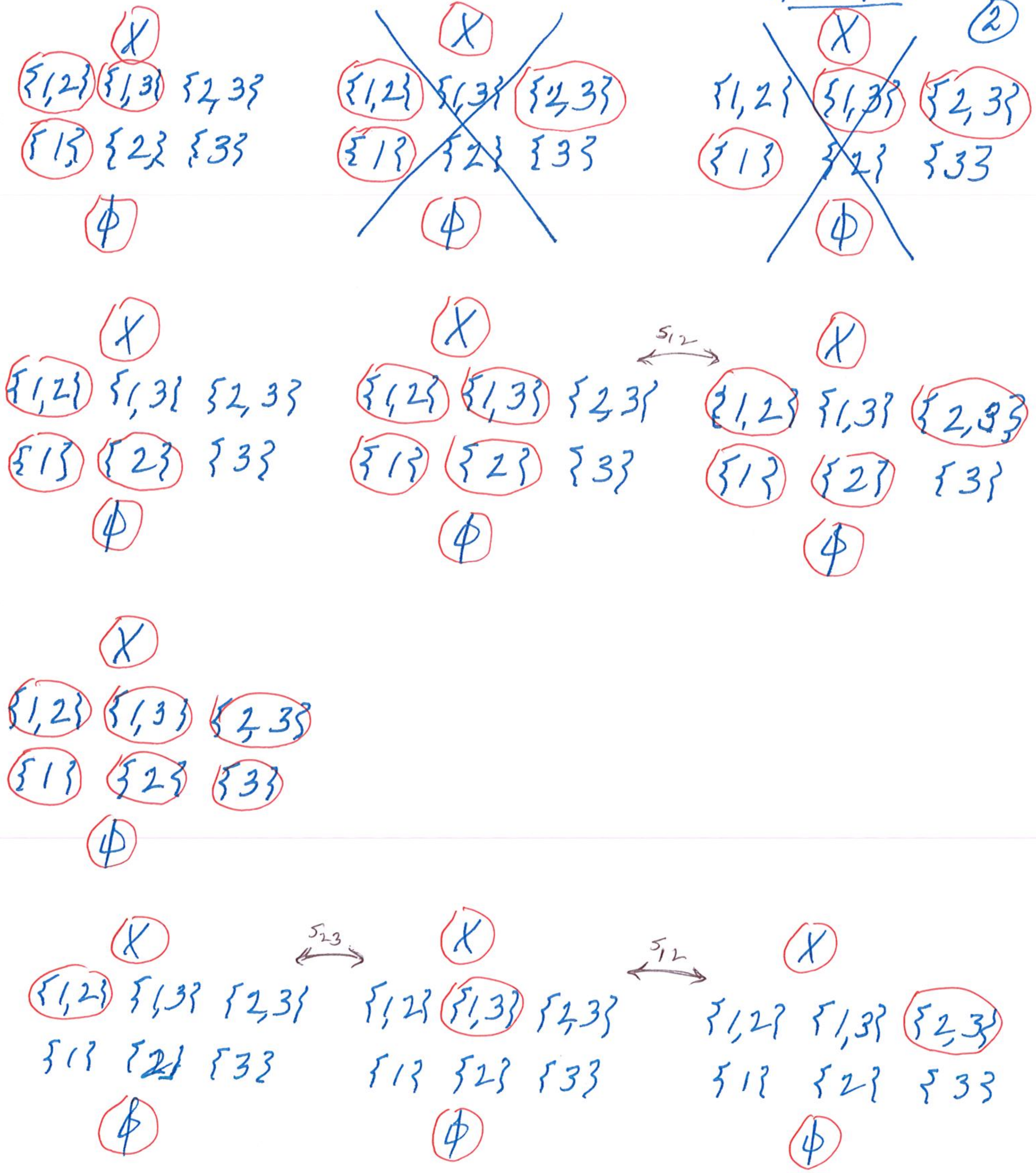

The corresponding preorders are

X
$\{1,2\}$ $\{1,3\}$ $\{2,3\}$
$\{1\}$ $\{2\}$ $\{3\}$
$\emptyset$

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\end{array}$$

X
$\{1,2\}$ $\{1,3\}$ $\{2,3\}$
$\{1\}$ $\{2\}$ $\{3\}$
$\emptyset$

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\end{array}$$

X
$\{1,2\}$ $\{1,3\}$ $\{2,3\}$
$\{1\}$ $\{2\}$ $\{3\}$
$\emptyset$

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\end{array}$$

X
$\{1,2\}$ $\{1,3\}$ $\{2,3\}$
$\{1\}$ $\{2\}$ $\{3\}$
$\emptyset$

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}\end{array}$$

X
$\{1,2\}$ $\{1,3\}$ $\{2,3\}$
$\{1\}$ $\{2\}$ $\{3\}$
$\emptyset$

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3\end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\end{array}$$

X

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$

$\{1\}$ $\{2\}$ $\{3\}$

ϕ

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}\right) & \\ 3 & & & \end{array}$$

X

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$

$\{1\}$ $\{2\}$ $\{3\}$

ϕ

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left(\begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}\right) & \\ 3 & & & \end{array}$$

X

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$

$\{1\}$ $\{2\}$ $\{3\}$

ϕ

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right) & \\ 3 & & & \end{array}$$

where a 1 in position (i,j) indicates $i \leq j$.

(5b) Let $(X, \le)$ be a preordered set.

Let $\mathcal{T}_X = \{U \subseteq X \mid$ if $x \in U$ and $y \in X$ and $x \le y$ then $y \in U\}$.

To show: $\mathcal{T}_X$ is a topology.

To show: (ba) $\phi \in \mathcal{T}_X$ and $X \in \mathcal{T}_X$

(bb) If $\mathcal{S} \subseteq \mathcal{T}_X$ then $\left(\bigcup_{U \in \mathcal{S}} U\right) \in \mathcal{T}_X$

(bc) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}_X$ then

$U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}_X$.

(ba) (baa) If $x \in \phi$ and $y \in X$ and $x \le y$ then $y \in \phi$ is vacuously satisfied.

(bab) If $x \in X$ and $y \in X$ and $x \le y$ then $y \in X$ is true.

(bb) Assume $\mathcal{S} \subseteq \mathcal{T}_X$. Let $A = \left(\bigcup_{U \in \mathcal{S}} U\right)$

To show: If $x \in A$ and $y \in X$ and $x \le y$ then $y \in A$.

Assume $x \in A$ and $y \in X$ and $x \le y$.

To show: $y \in A$

Since $x \in A$ there exist $U \in \mathcal{S}$ with $x \in U$.

Since $x \in U$ and $x \le y$ and $U \in \mathcal{T}_X$ then $y \in U$.

So $y \in \left(\bigcup_{U \in \mathcal{S}} U\right)$. So $y \in A$.

So $A \in \mathcal{T}_X$

(bc) Assume $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}_X$

Let $A = U_1 \cap \cdots \cap U_\ell$.

To show: If $x \in A$ and $y \in X$ and $x \le y$ then $x \in A$.

Assume $x \in A$ and $y \in X$ and $x \le y$.

To show: $y \in U_1 \cap \cdots \cap U_\ell$.

Assume $j \in \{1, \ldots, \ell\}$.

Since $x \in U_j$ and $x \le y$ and $U_j \in \mathcal{T}_X$ then $y \in U_j$.

So $y \in U_1 \cap \cdots \cap U_\ell$.

So $\mathcal{T}_X$ is a topology.

(5c) Let $(Y, \mathcal{T})$ be a topological space.

To show: $\le$ is a preorder.

To show: (ca) If $a \in Y$ then $a \le a$

(cb) If $a, b, c \in Y$ and $a \le b$ and $b \le c$ then $a \le c$.

(ca) Assume $a \in Y$.

To show: $a \le a$

To show: $a \in \overline{\{a\}}$

Since $\overline{\{a\}} \supseteq \{a\}$ then $a \in \overline{\{a\}}$.

So $a \le a$.

(cb) Assume $a, b, c \in Y$ and $a \le b$ and $b \le c$.

To show: $a \le c$.

To show: $a \in \overline{\{c\}}$

We know: $a \in \overline{\{b\}}$ and $b \in \overline{\{c\}}$

Since $\{b\} \subseteq \overline{\{c\}}$ and $\overline{\{c\}}$ is closed then $\overline{\{b\}} \subseteq \overline{\{c\}}$

So $a \in \overline{\{b\}} \subseteq \overline{\{c\}}$

So $a \in \overline{\{c\}}$

So $a \le c$.

So $\le$ is a preorder. //

(5d) Assume $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is continuous.

To show: $f: F(X)\to F(Y)$ is monotone.

To show: If $x_1,x_2\in X$ and $x_1\le x_2$ then $f(x_1)\le f(x_2)$.

Assume $x_1,x_2\in X$ and $x_1\le x_2$.

To show: $f(x_1)\le f(x_2)$

To show: $f(x_1)\in\overline{\{f(x_2)\}}$

To show: $x_1\in f^{-1}(\overline{\{f(x_2)\}})$

We know: $x_2\in f^{-1}(\overline{\{f(x_2)\}})$ and $f^{-1}(\overline{\{f(x_2)\}})$ is closed, since f is continuous.

So $f^{-1}(\overline{\{f(x_2)\}})\supseteq\overline{\{x_2\}}$

Since $x_1\le x_2$ then $x_1\in\overline{\{x_2\}}\subseteq f^{-1}(\overline{\{f(x_2)\}})$.

So $f(x_1)\in\overline{\{f(x_2)\}}$.

So $f(x_1)\le f(x_2)$.

So f is monotone. //

(5e) Let $(X, \le_X)$ and $(Y, \le_Y)$ be preordered sets. Let

$\mathcal{T}_X = \{U \subseteq X \mid \text{if } x \in U \text{ then } U_x \subseteq U\}$,

where $U_x = \{a \in X \mid a \ge_X x\}$.

<u>Aside</u>: Assume $z \in U_X$. Let $b \in U_z$. So $b \ge z$ and $z \ge x$. So $b \in U_X$. Thus $U_z \subseteq U_x$.
So $U_x \in \mathcal{T}_X$.

Let $\mathcal{T}_Y = \{V \subseteq Y \mid \text{if } y \in V \text{ then } V_y \subseteq V\}$,
where $V_y = \{b \in Y \mid b \ge_Y y\}$.

Assume $f: X \to Y$ satisfies:
if $x_1, x_2 \in X$ and $x_1 \le_X x_2$ then $f(x_1) \le_Y f(x_2)$.

To show: $f: X \to Y$ is continuous.

To show: If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.

Assume $V \in \mathcal{T}_Y$

To show $f^{-1}(V) \in \mathcal{T}_X$

To show: If $x \in f^{-1}(V)$ then $U_x \subseteq f^{-1}(V)$.

Assume $x \in f^{-1}(V)$

Then $f(x) \in V$.

To show: $U_x \subseteq f^{-1}(V)$

To show: If $a \in U_x$ then $f(a) \in V$.

Assume $a \in U_x$.

Then $a \geq x$.

So $f(a) \geq f(x)$.

So $f(a) \in U_{f(x)}$.

Since $f(x) \in V$ ~~then~~ and $V \in \mathcal{T}_y$ then $U_{f(x)} \subseteq V$.

So $f(a) \in U_{f(x)} \subseteq V$.

So $a \in f^{-1}(V)$

So $U_x \subseteq f^{-1}(V)$.

So $f^{-1}(V) \in \mathcal{T}_x$

So f is continuous. //

(5f) Let $(X, \le)$ be a preordered set.

Let $\mathcal{T} = \{U \subseteq X \mid$ if $x \in U$ and $y \in X$ and $x \le y$ then $y \in U\}$

$= \{U \subseteq X \mid$ if $x \in U$ then $N_x \subseteq U\}$,

where $N_x = \{y \in X \mid y \ge x\}$.

Then $\mathcal{G}(X, \le) = (X, \mathcal{T})$.

Define $\precsim$ on X by

$z \precsim y$ if $z \in \overline{\{y\}}$

To show: (a) If $z \precsim y$ then $z \le y$

(b) If $z \le y$ then $z \precsim y$

(a) Assume $z \precsim y$.

Then $z \in \overline{\{y\}}$

So z is a close point to y.

If $a \in N_z$ then $N_a \subseteq N_z$ since if $b \ge a$ and $a \ge z$ then $b \ge z$.

So $N_z \in \mathcal{T}$ and $N_z \in \mathcal{N}(z)$.

So $N_z \cap \{y\} \ne \emptyset$

So $y \in N_z$.

So $y \ge z$.

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(b) Assume $z \le y$.

To show: $z \preccurlyeq y$.

To show: $z \in \overline{\{y\}}$

To show: z is a close point to $\{y\}$

To show: If $N \in \mathcal{N}(z)$ then $N \cap \{y\} \neq \phi$.

Assume $N \in \mathcal{N}(z)$.

Then $z \in N$ and there is $U \in \mathcal{T}$ with $z \in U \subseteq N$.

So $N_z \subseteq U \subseteq N$.

Since $y \ge z$ then $y \in N_z \subseteq N$.

So $N \cap \{y\} \neq \phi$.

So z is a close point to $\{y\}$.

So $z \in \overline{\{y\}}$

So $z \preccurlyeq y$. //

(5g) Let $X = \mathbb{R}$ with the standard topology $\mathcal{T}$.

Let $y \in X$. Then $\overline{\{y\}} = \{y\}$ (since $\{y\}^c = (-\infty, y) \cup (y, \infty)$ is open).

So the preorder on X is defined by

$$y \leq y \quad \text{and} \quad x \not\leq y \text{ if } x \neq y. \qquad (*)$$

So $\mathcal{F}(X, \mathcal{T}) = (X, \leq)$, with $\leq$ as in $(*)$.

Define
$$\mathcal{T}' = \left\{ U \subseteq \mathbb{R} \;\middle|\; \begin{array}{l} \text{if } x \in U \text{ and } y \in \mathbb{R} \\ \text{and } x \leq y \text{ then } y \in U \end{array} \right\}$$

$$= \{U \subseteq \mathbb{R}\}$$

since any subset $U \subseteq \mathbb{R}$ satisfies

if $x \in U$ and $y \in \mathbb{R}$ and $x \leq y$ then $y \in U$

because the <u>only</u> $y \in \mathbb{R}$ with $y \geq x$ is $y = x$.

So $\mathcal{G}(\mathcal{F}(X, \mathcal{T})) = (X, \mathcal{T}')$ and $\mathcal{T}'$ is the discrete topology on $\mathbb{R}$ (every set is open).

$\mathcal{T}' \neq \mathcal{T}$ since $\{2\}$ is open in $\mathcal{T}'$ and $\{2\}$ is not open in $\mathcal{T}$.