(2a) Let $X = \{0,1\}$ and $\mathcal{T}_X = \{\phi, \{0\}, \{0,1\}\}$

Let $Y = \{0,1\}$ and $\mathcal{T}_Y = \{\phi, \{1\}, \{0,1\}\}$.

Then $X \times Y = \{(0,0), (0,1), (1,0), (1,1)\}$, the set of rectangles is $\mathcal{R} = \left\{ \begin{array}{l} \phi, \{(0,1)\}, \{(0,0), (0,1)\} \\ \{\cancel{(0,0)}, (0,1), (1,1)\}, X \times X \end{array} \right\}$

and the product topology is

$$\mathcal{T}_{X \times Y} = \left\{ \begin{array}{l} \phi, \{(0,1)\}, \{(0,0), (0,1)\}, \{(0,1), (1,1)\}, \\ \{(0,0), (0,1), (1,1)\}, X \times X \end{array} \right\}$$

Then $Z = \{(0,0), (0,1), (1,1)\}$ is an element of $\mathcal{T}_{X \times Y}$ that is not a rectangle.

<u>Another example</u> (which provides additional depth of insight):

Let $X = \mathbb{R}$ with the standard topology

$Y = \mathbb{R}$ with the standard topology.

Then $X \times Y = \mathbb{R}^2$ and

$$B_1(0) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

is an open set in $\mathbb{R}^2$.

If there exists U and V with $B_1(0) = U \times V$ then

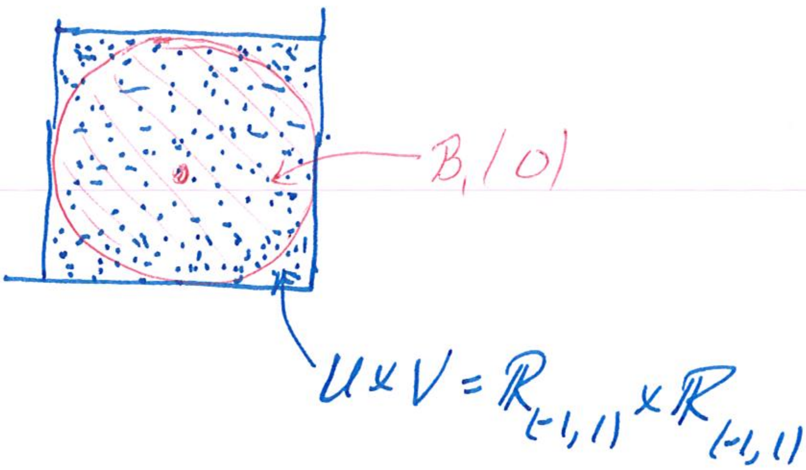
$U = \{$ x-coordinates of points in $B_1(0)\}$

$= \mathbb{R}_{(-1,1)}$, and

$V = \{$ y-coordinates of points in $B_1(0)\}$

$= \mathbb{R}_{(-1,1)}$.

But $B_1(0) \neq \mathbb{R}_{(-1,1)} \times \mathbb{R}_{(-1,1)}$



(2b) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

Let $\mathcal{T}_{X\times Y}$ be the product topology on $X \times Y$:

$$\mathcal{T}_{X\times Y} = \{\text{unions of open rectangles}\}$$

$$= \left\{ Z \subseteq X\times Y \;\middle|\; \begin{matrix}\text{there exists } \mathcal{S} \subseteq \mathcal{R} \text{ such that} \\ Z = \bigcup_{U\times V \in \mathcal{S}} (U \times V)\end{matrix} \right\}$$

where $\mathcal{R} = \{ U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \}$

By definition, if $(x,y) \in X \times Y$ then

$$\mathcal{N}((x,y)) = \left\{ N \subseteq X \times Y \;\middle|\; \begin{matrix}\text{there exists } Z \in \mathcal{T}_{X\times Y} \\ \text{with } (x,y) \in Z \text{ and } Z \subseteq N\end{matrix} \right\}$$

Let $$\mathcal{P} = \left\{ W \subseteq X \times Y \;\middle|\; \begin{matrix}\text{there exists } U \in \mathcal{N}(x) \text{ and } V \in \mathcal{N}(y) \\ \text{with } U \times V \subseteq W\end{matrix} \right\}$$

To show: $\mathcal{N}((x,y)) = \mathcal{P}$.

To show: (ba) $\mathcal{N}((x,y)) \subseteq \mathcal{P}$

(bb) $\mathcal{N}((x,y)) \supseteq \mathcal{P}$

(ba) To show: If $N \in \mathcal{N}((x,y))$ then $N \in \mathcal{P}$.

Assume $N \in \mathcal{N}((x,y))$

To show: $N \in \mathcal{P}$

Since $N \in \mathcal{N}((x,y))$ then there exists $Z \in \mathcal{T}_{X\times Y}$ with $(x,y) \in Z$ and $Z \subseteq N$.

So there exists $\mathcal{S} \subseteq \mathcal{R}$ with $Z = \bigcup_{U\times V \in \mathcal{S}} U\times V$

So there exists $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that $(x,y) \in U\times V$ and $U \times V \subseteq Z$.

So $x \in U$ and $y \in V$ and $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$.

So $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ and $U\times V \subseteq Z \subseteq N$.

So $N \in \mathcal{P}$.

(bb) To show: $\mathcal{P} \subseteq \mathcal{N}((x,y))$.

Assume $W \in \mathcal{P}$

To show: $W \in \mathcal{N}((x,y))$.

Since $W \in \mathcal{P}$ then there exists $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U\times V \subseteq W$.

So there exists $A \in \mathcal{T}_X$ with $x \in A$ and $A \subseteq U$, and there exists $B \in \mathcal{T}_Y$ with $y \in B$ and $B \subseteq V$.

So $(x,y) \in A \times B \subseteq U\times V \subseteq W$.

Since $A \in \mathcal{T}_X$ and $B \in \mathcal{T}_Y$ then $A\times B \in \mathcal{T}_{X\times Y}$.

Let $Z = A \times B$.

Then $Z \in \mathcal{T}_{X \times Y}$ and $(x, y) \in Z$ and $Z \subseteq W$.

So $W \in \mathcal{N}((x, y))$.

So $\mathcal{P} \subseteq \mathcal{N}((x, y))$.

So $\mathcal{P} = \mathcal{N}((x, y))$.

(2c) Let X and Y be topological spaces.
Let $A \subseteq X$ and $B \subseteq Y$. Show that $\bar{A} \times \bar{B} = \overline{A \times B}$.

<u>Proof</u> To show: (a) $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$

(b) $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$

(a) Assume $(x,y) \in \bar{A} \times \bar{B}$

To show: $(x,y) \in \overline{A \times B}$

To show: (x,y) is a close point of $A \times B$.

Let N be a neighborhood of (x,y) in $X \times Y$.

By the definition of the product topology on $X \times X$ there exist

N_x, a neighborhood of x on X,

and N_y, a neighborhood of y in Y,

such that $N_x \times N_y \subseteq N$.

Since $x \in \bar{A}$ there exists $a \in A$ with $a \in N_x$.

Since $y \in \bar{B}$ there exists $b \in B$ with $b \in N_y$.

So $(a,b) \in N_x \times N_y \subseteq N$ and $(a,b) \in A \times B$.

So (x,y) is a close point of $A \times B$.

So $(x,y) \in \overline{A \times B}$

So $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$

(b) To show: $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$

Assume $(x,y) \in \overline{A \times B}$

To show: $(x,y) \in \overline{A} \times \overline{B}$.

Let N_x be a neighborhood of $x \in X$ and let N_y be a neighborhood of $y \in Y$.

Then $N_x \times N_y$ is a neighborhood of $(x,y) \in X \times Y$.

Since (x,y) is a close point of $A \times B$,

there exists $(a,b) \in A \times B$ with $(a,b) \in N_x \times N_y$.

So $a \in N_x$ and $b \in N_y$ and $a \in A$ and $b \in B$.

So x is a close point of A and

y is a close point of B.

So $x \in \overline{A}$ and $y \in \overline{B}$

So $(x,y) \in \overline{A} \times \overline{B}$.

(2d) Assume X and Y are path connected

To show: $X \times Y$ is path connected.

To show: If $(x_1, y_1), (x_2, y_2) \in X \times Y$ then there exists a path $p: [0,1] \to X \times Y$ connecting (x_1, y_1) and (x_2, y_2).

Assume $(x_1, y_1) \in X \times Y$ and $(x_2, y_2) \in X \times Y$.

Since X and Y are path connected we know that there exist continuous functions

$$p_1: [0,1] \to X \quad \text{and} \quad p_2: [0,1] \to Y$$

with
$$p_1(0) = x_1 \quad \text{and} \quad p_2(0) = y_1$$
$$p_1(1) = x_2 \qquad\quad p_2(1) = y_2.$$

To show: There exists a continuous function $p: [0,1] \to X \times Y$ with $p(0) = (x_1, y_1)$ and $p(1) = (x_2, y_2)$.

Let $p: [0,1] \to X \times Y$ be given by $p(t) = (p_1(t), p_2(t))$

Then $p(0) = (p_1(0), p_2(0)) = (x_1, y_1)$,

$p(1) = (p_1(1), p_2(1)) = (x_2, y_2)$, ~~and~~

To show: p is continuous

To show: If V is open in $X \times Y$ then
$p^{-1}(V)$ is open in $[0,1]$.

Assume V is open in $X \times Y$.
To show: $p^{-1}(V)$ is open in $[0,1]$.

To show: If $c \in p^{-1}(V)$ then c is an interior point of $p^{-1}(V)$.

Assume $c \in p^{-1}(V)$.
To show: c is an interior point of $p^{-1}(V)$.
To show: There exists a neighborhood $N \in \mathcal{N}(c)$ such that $N \subseteq p^{-1}(V)$.

Let $p(c) = (x,y)$.
Let $V_x \in \mathcal{N}(x)$ and $V_y \in \mathcal{N}(y)$ such that
$V_x \times V_y \subseteq V$ (definition of product topology on $X \times Y$.
Let $N = \cancel{p^{-1}(V_x \times V_y)}$. $p_1^{-1}(V_x) \cap p_2^{-1}(V_y)$.
Since p_1 is continuous $p_1^{-1}(V_x)$ is open,
since p_2 is continuous $p_2^{-1}(V_y)$ is open.
Then $c \in p_1^{-1}(V_x) \cap p_2^{-1}(V_y) \subseteq p^{-1}(V_x \times V_y) \subseteq p^{-1}(V)$.
So c is an interior point of $p^{-1}(V)$
So $p^{-1}(V)$ is open.
So p is continuous