(1a) Let A and B be bounded subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$.
Show that

$$\mathrm{diam}(A \cup B) \leq \mathrm{diam}(A) + \mathrm{diam}(B).$$

Solution The definition of diam(A) is

$$\mathrm{diam}(A) = \sup\{d(x,y) \mid x, y \in A\}.$$

Assume $A \subseteq X$ and $B \subseteq X$ and $A \cap B \neq \emptyset$ and $\mathrm{diam}(A) < \infty$ and $\mathrm{diam}(B) < \infty$.

To show: $\mathrm{diam}(A \cup B) \leq \mathrm{diam}(A) + \mathrm{diam}(B)$

To show: $\mathrm{diam}(A) + \mathrm{diam}(B)$ is an upper bound of $\{d(x,y) \mid x, y \in A \cup B\}$.

To show: If $x, y \in A \cup B$ then $d(x,y) \leq \mathrm{diam}(A) + \mathrm{diam}(B)$

Assume $x, y \in A \cup B$.

Case 1: $x, y \in A$. Then

$$d(x,y) \leq \mathrm{diam}(A) \leq \mathrm{diam}(A) + \mathrm{diam}(B)$$

Case 2: $x, y \in B$. Then

$$d(x,y) \leq \mathrm{diam}(B) \leq \mathrm{diam}(A) + \mathrm{diam}(B).$$

<u>Case 3</u>: $x \in A$ and $y \in B$. Let $z \in A \cap B$. Then

$$d(x,y) \le d(x,z) + d(z,y) \le \operatorname{diam}(A) + \operatorname{diam}(B).$$

<u>Case 4</u>: $x \in B$ and $y \in A$. Let $z \in A \cap B$. Then

$$d(x,y) \le d(x,z) + d(z,y) \le \operatorname{diam}(B) + \operatorname{diam}(A).$$

So $\operatorname{diam}(A) + \operatorname{diam}(B)$ is an upper bound of

$$\{(x,y) \mid x,y \in A \cup B\}.$$

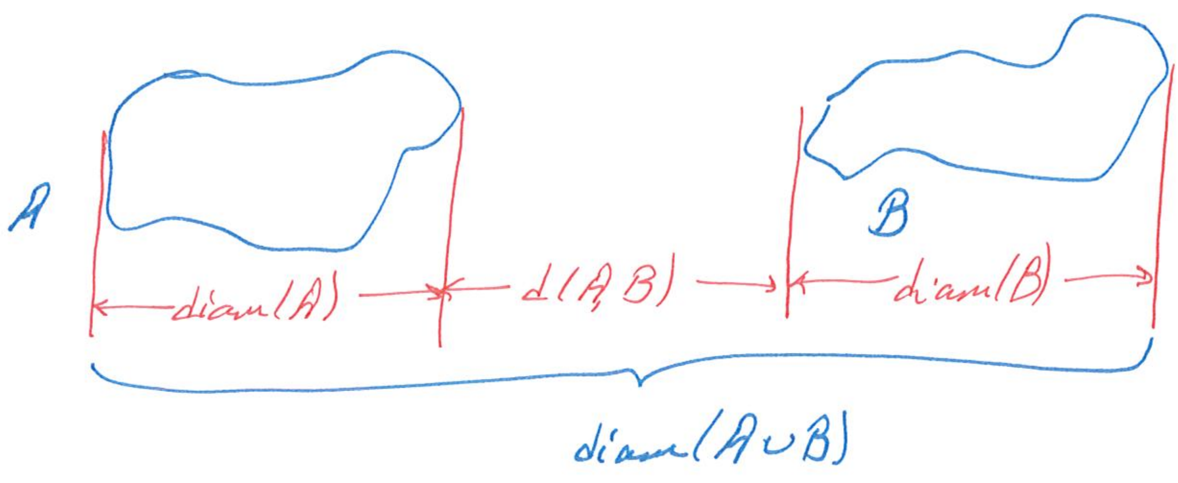
So $\operatorname{diam}(A \cup B) \le \operatorname{diam}(A) + \operatorname{diam}(B)$

If $A \cap B = \phi$ then we expect

$$\operatorname{diam}(A \cup B) = \operatorname{diam}(A) + d(A,B) + \operatorname{diam}(B)$$

where

$$d(A,B) = \inf\{d(x,y) \mid x \in A, y \in B\}$$



(1b) Prove that $\bar{A} = \{x \in X \mid d(x, A) = 0\}$

To show: (a) $\{x \in X \mid d(x, A) = 0\} \subseteq \bar{A}$

(b) $\bar{A} \subseteq \{x \in X \mid d(x, A) = 0\}$.

(a) Assume $x \in X$ and $d(x, A) = 0$

To show: $x \in \bar{A}$

Let N be a neighborhood of x in X.

Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(x) \subseteq N$.

Since $d(x, A) = \inf\{d(x, a) \mid a \in A\} = 0$,

there exists $a \in A$ such that $d(x, a) < \varepsilon$

Then $a \in B_\varepsilon(x) \subseteq N$ and $a \in A$.

So x is a close point of A.

So $\{x \in X \mid d(x, A) = 0\} \subseteq \bar{A}$.

(b) To show: $\bar{A} \subseteq \{x \in X \mid d(x, A) = 0\}$

Let $x \in \bar{A}$.

So x is a close point of A.

To show: $d(x, A) = 0$

Let $\varepsilon \in \mathbb{R}_{>0}$

Then $B_\varepsilon(x)$ is a neighborhood of $x \in X$.

Since x is a close point of A

there exists $a \in A$ such that $a \in B_\varepsilon(x)$.

So $d(x,a) < \varepsilon$.

So $d(x,A) < \varepsilon$ for all $\varepsilon \in \mathbb{R}_{>0}$.

So $d(x,A) = 0$.

So $x \in \{x \in X \mid d(x,A) = 0\}$.

So $\overline{A} \subseteq \{x \in X \mid d(x,A) = 0\}$

Thus $\overline{A} = \{x \in X \mid d(x,A) = 0\}$.

(1c) Show that if $x, y \in X$ then

$$|d(x,A) - d(y,A)| \le d(x,y)$$

Assume $x, y \in X$.

To show: (a) $d(x,A) - d(y,A) \le d(x,y)$
(b) $-(d(x,A) - d(y,A)) \le d(x,y)$.

(a) Since $d(x,A)$ is a lower bound of $\{d(x,a) \mid a \in A\}$

if $a \in A$ then $d(x,A) \le d(x,a)$

Using $d(x,a) \le d(x,y) + d(y,a)$,

if $a \in A$ then $d(x,A) \le d(x,y) + d(y,a)$.

So $d(x,A)$ is a lower bound of $\{d(x,y) + d(y,a) \mid a \in A\}$.

Since $d(x,y) + d(y,A)$ is the <u>greatest</u> lower bound of

$\{d(x,y) + d(y,a) \mid a \in A\}$ then

$$d(x,A) \le d(x,y) + d(y,A).$$

So $d(x,A) - d(y,A) \le d(x,y)$.

So $d(y,A) - d(x,A) \le d(y,x) = d(x,y)$.

So $-(d(x,A)-d(y,A)) \leq d(x,y)$.

So $d(x,A)-d(y,A) \leq d(x,y)$ and $-(d(x,A)-d(y,A)) \leq d(x,y)$.

So $|d(x,A)-d(y,A)| \leq d(x,y)$.

(1d) Let $f: X \to \mathbb{R}$ be given by $f(x) = d(x, A)$.

Show that f is continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ and $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in X$ and $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$

Let $\delta = \varepsilon$.

To show: If $y \in X$ and $d(x,y) < \delta$ then

$d(f(x), f(y)) < \varepsilon$

Assume $y \in X$ and $d(x,y) < \delta$.

To show: $d(f(x), f(y)) < \varepsilon$.

By (1c),

$$d(f(x), f(y)) = |d(x,A) - d(y,A)| \le d(x,y) < \delta = \varepsilon.$$

So f is continuous

(1e) Assume $x \notin \bar{A}$ and let $U = \{y \in X \mid d(y,A) < d(x,A)\}$

Show that (a) $x \in U$

(b) U is open

(c) $\bar{A} \subseteq U$.

(a) Let $D = d(x,A)$.

Since $x \notin \bar{A}$ and, by part (a), $\bar{A} = \{y \in X \mid d(y,A) = 0\}$

then $d(x,A) \neq 0$.

So $D \neq 0$.

We know $U = \{y \in X \mid d(y,A) < D\}$

Since $d(x,A) = D$ then $x \notin U$.

(b) Since $U \in f^{-1}(\mathbb{R}_{<D}) = f^{-1}((-\infty, D))$

and f is continuous then U is open.

(c) By (1b),

$\bar{A} = \{y \in X \mid d(y,A) = 0\} \subseteq \{y \in X \mid d(y,A) < D\} = U$.

So $\bar{A} \subseteq U$. //